# Random networks with sublinear preferential attachment: Degree evolutions

by

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***Summary.*** We define a dynamic model of random networks, where new vertices are connected to old ones with a probability proportional to a sublinear function of their degree. We first give a strong limit law for the empirical degree distribution, and then have a closer look at the temporal evolution of the degrees of individual vertices, which we describe in terms of large and moderate deviation principles. Using these results, we expose an interesting phase transition: in cases of *strong* preference of large degrees, eventually a single vertex emerges forever as vertex of maximal degree, whereas in cases of *weak* preference, the vertex of maximal degree is changing infinitely often. Loosely speaking, the transition between the two phases occurs in the case when a new edge is attached to an existing vertex with a probability proportional to the root of its current degree.

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# 1 Introduction

## 1.1 Motivation

Dynamic random graph models, in which new vertices prefer to be attached to vertices with higher degree in the existing graph, have proved to be immensely popular in the scientific literature recently. The two main reasons for this popularity are, on the one hand, that these models can be easily defined and modified, and can therefore be calibrated to serve as models for social networks, collaboration and interaction graphs, or the web graph. On the other hand, if the attachment probability is approximately proportional to the degree of a vertex, the dynamics of the model can offer a credible explanation for the occurrence of power law degree distributions in large networks.

The philosophy behind these preferential attachment models is that growing networks are built by adding nodes successively. Whenever a new node is added it is linked by edges to one or more existing nodes with a probability proportional to a function $f$ of their degree. This function $f$, called *attachment rule*, or sometimes *weight function*, determines the qualitative features of the dynamic network.

The heuristic characterisation does not amount to a full definition of the model, and some clarifications have to be made, but it is generally believed that none of these crucially influence the long time behaviour of the model.

It is easy to see that in the general framework there are *three* main regimes:

- the *linear* regime, where $f(k) \asymp k$;
- the *superlinear* regime, where $f(k) \gg k$;
- the *sublinear* regime, where $f(k) \ll k$.

The linear regime has received most attention, and a major case has been introduced in the much-cited paper Barabási and Albert (1999). There is by now a substantial body of rigorous mathematical work on this case. In particular, it is shown in Bollobás et al. (2001), Móri (2002) that the empirical degree distribution follows an asymptotic power law and in Móri (2005) that the maximal degree of the network is growing polynomially of the same order as the degree of the first node.

In the superlinear regime the behaviour is more extreme. In Oliveira and Spencer (2005) it is shown that a dominant vertex emerges, which attracts a positive proportion of all future edges. Asymptotically, after $n$ steps, this vertex has degree of order $n$, while the degrees of all other vertices are bounded. In the most extreme cases eventually all vertices attach to the dominant vertex.

In the linear and sublinear regimes Rudas et al. (2007) find almost sure convergence of the empirical degree distributions. In the linear regime the limiting distribution obeys a power law, whereas in the sublinear regime the limiting distributions are stretched exponential distributions. Apart from this, there has not been much research so far in the sublinear regime, which is the main concern of the present article, though we include the linear regime in most of our results.

Specifically, we discuss a preferential attachment model where new nodes connect to a random number of old nodes, which in fact is quite desirable from the modelling point of view. More precisely, the node added in the $n$th step is connected independently to any old one with probability $f(k)/n$, where $k$ is the (in-)degree of the old node. We first determine the asymptotic degree distribution, see Theorem 1.1, and find a result which is in line with that of Rudas et al. (2007). The result implies in particular that, if $f(k) = (k+1)^\alpha$ for $0 \leqslant \alpha < 1$, then the asymptotic degree distribution $(\mu_k)$ satisfies

$$\log \mu_k \sim -\tfrac{1}{1-\alpha}\, k^{1-\alpha},$$

showing that power law behaviour is limited to the linear regime. Under the assumption that the strength of the attachment preference is sufficiently weak, we give very fine results about the probability that the degree of a fixed vertex follows a given increasing function, see Theorem 1.10 and Theorem 1.12. These large and moderate deviation results, besides being of independent interest, play an important role in the proof of our main result. This result describes an interesting dichotomy about the behaviour of the vertex of maximal degree, see Theorem 1.7:

- *The strong preference case:* If $\sum_n 1/f(n)^2 < \infty$, then there exists a single dominant vertex –called *persistent hub*– which has maximal degree for all but finitely many times. However, only in the linear regime the number of new vertices connecting to the dominant vertex is growing polynomially in time.

- *The weak preference case:* If $\sum_n 1/f(n)^2 = \infty$, then there is almost surely *no* persistent hub. In particular, the index, or time of birth, of the current vertex of maximal degree is a function of time diverging to infinity in probability. In Theorem 1.15 we provide asymptotic results for the index and degree of this vertex, as time goes to infinity.

A rigorous definition of the model is given in Section 1.2, and precise statements of all the principal results follow in Section 1.3. At the end of that section, we also give a short overview over the further parts of this paper.

### 1.2 Definition of the model

We now explain how precisely we define our preferential attachment model given a monotonically increasing *attachment rule* $f\colon \{0,1,2,\dots\} \longrightarrow (0,\infty)$ with $f(n) \leqslant n+1$ for all $n \in \mathbb{Z}_+ := \{0,1,\dots\}$. At time $n=1$ the network consists of a single vertex (labeled 1) without edges and for each $n \in \mathbb{N}$ the graph evolves in the time step $n \to n+1$ according to the following rule

- add a new vertex (labeled $n+1$) and
- insert for each old vertex $m$ a directed edge $n+1 \to m$ with probability

$$\frac{f(\text{indegree of } m \text{ at time } n)}{n}.$$

The new edges are inserted independently for each old vertex. Note that the assumptions imposed on $f$ guarantee that in each evolution step the probability for adding an edge is smaller or equal to 1. Formally we are dealing with a directed network, but indeed, by construction, all edges are pointing from the younger to the older vertex, so that the directions can trivially be recreated from the undirected (labeled) graph.

There is one notable change to the recipe given in Krapivsky and Redner (2001): We do not add one edge in every step but a random number, a property which is actually desirable in most applications. Given the graph after attachment of the $n$th vertex, the expected number of edges added in the next step is

$$\frac{1}{n}\sum_{m=1}^{n} f\big(\text{indegree of } m \text{ at time } n\big)\,.$$

This quantity converges, as $n \to \infty$ almost surely to a deterministic limit $\lambda$, see Theorem 1.1. Moreover, the law of the number of edges added is asymptotically Poissonian with parameter $\lambda$. Observe that the *out*degree of every vertex remains unchanged after the step in which the vertex was created. Hence our principal interest when studying the asymptotic evolution of degree distributions is in the *in*degrees.

### 1.3 Presentation of the main results

We denote by $\mathcal{Z}[m,n]$, for $m,n \in \mathbb{N}$, $m \leqslant n$, the indegree of the $m$-th vertex after the insertion of the $n$-th vertex, and by $X_k(n)$ the proportion of nodes of indegree $k \in \mathbb{Z}_+$ at time $n$, that is

$$X_k(n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{Z}[i,n]=k\}}.$$

Moreover, denote $\mu_k(n) = \mathbb{E}X_k(n)$, $X(n) = (X_k(n)\colon k \in \mathbb{Z}_+)$, and $\mu(n) = (\mu_k(n)\colon k \in \mathbb{Z}_+)$.

**Theorem 1.1** (Asymptotic empirical degree distribution)**.**

*(i) Let*

$$\mu_k = \frac{1}{1+f(k)}\prod_{l=0}^{k-1}\frac{f(l)}{1+f(l)} \qquad \text{for } k \in \mathbb{Z}_+,$$

*which is a sequence of probability weights. Then, almost surely,*

$$\lim_{n\to\infty} X(n) = \mu$$

*in total variation norm.*

*(ii) If $f$ satisfies $f(k) \leqslant \eta k + 1$ for some $\eta \in (0,1)$, then the conditional distribution of the outdegree of the $(n+1)$st incoming node (given the graph at time $n$) converges almost surely in variation topology to the Poisson distribution with parameter $\lambda := \langle \mu, f\rangle$.*

**Remark 1.2.** The asymptotic degree distribution coincides with that in the random tree model introduced in Krapivsky and Redner (2001) and studied by Rudas et al. (2007), if $f$ is chosen as an appropriate multiple of their weight function. This is strong evidence that these models show the same qualitative behaviour, and that our further results hold *mutatis mutandis* for preferential attachment models in which new vertices connect to a fixed number of old ones.

**Example 1.3.** Suppose $f(k) \sim \gamma k^{\alpha}$, for $0 < \alpha < 1$ and $\gamma > 0$, then a straight forward analysis yields that

$$\log \mu_k \sim -\sum_{l=1}^{k+1} \log\left(1 + l^{-\alpha}\right) \sim -\tfrac{1}{\gamma}\,\tfrac{1}{1-\alpha}\,k^{1-\alpha}\,.$$

Hence the asymptotic degree distribution has stretched exponential tails.

In order to analyse the network further, we scale the time as well as the way of counting the indegree. To the *original* time $n \in \mathbb{N}$ we associate an *artificial* time

$$\Psi(n) := \sum_{m=1}^{n-1} \frac{1}{m} \sim \log n,$$

and to the *original* degree $j \in \mathbb{Z}_+$ we associate the *artificial* degree

$$\Phi(j) := \sum_{k=0}^{j-1} \frac{1}{f(k)}.$$

An easy law of large numbers illustrates the role of these scalings.

**Proposition 1.4** (Law of large numbers)**.** *For any fixed vertex labeled $m \in \mathbb{N}$, we have that*

$$\lim_{n\to\infty} \frac{\Phi(\mathcal{Z}[m,n])}{\Psi(n)} = 1 \qquad \textit{almost surely}.$$

**Remark 1.5.** Since $\Psi(n) \sim \log n$, we conclude that for any $m \in \mathbb{N}$, almost surely,

$$\Phi(\mathcal{Z}[m,n]) \sim \log n \text{ as } n \to \infty.$$

In particular, we get for an attachment rule $f$ with $f(n) \sim \gamma n$ and $\gamma \in (0,1]$, that $\Phi(n) \sim \frac{1}{\gamma} \log n$ which implies that

$$\log \mathcal{Z}[m,n] \sim \log n^\gamma, \text{ almost surely.}$$

Furthermore, an attachment rule with $f(n) \sim \gamma n^\alpha$ for $\alpha < 1$ and $\gamma > 0$ leads to

$$\mathcal{Z}[m,n] \sim (\gamma(1-\alpha)\log n)^{\frac{1}{1-\alpha}}.$$

We denote by $\mathbb{T} := \{\Psi(n) \colon n \in \mathbb{N}\}$ the set of artificial times, and by $\mathbb{S} := \{\Phi(j) : j \in \mathbb{Z}_+\}$ the set of artificial degrees. From now on, we refer by *time* to the artificial time, and by *(in-)degree* to the artificial degree. Further, we introduce a new real-valued process $(Z[s,t])_{s\in\mathbb{T},t\geqslant 0}$ via

$$Z[s,t] := \Phi(\mathcal{Z}[m,n]) \quad \text{if } s = \Psi(m),\ t = \Psi(n) \text{ and } m \leqslant n,$$

and extend the definition to arbitrary $t$ by letting $Z[s,t] := Z[s, s \vee \max(\mathbb{T}\cap[0,t])]$. For notational convenience we extend the definition of $f$ to $[0,\infty)$ by setting $f(u) := f(\lfloor u \rfloor)$ for all $u \in [0,\infty)$, and linearly interpolate $\Phi$ at the breakpoints $\mathbb{S}$ so that

$$\Phi(u) = \int_0^u \frac{1}{f(v)}\, dv.$$

We denote by $\mathcal{L}[0,\infty)$ the space of càdlàg functions $x \colon [0,\infty) \to \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of $[0,\infty)$.

**Proposition 1.6** (Central limit theorem)**.** *In the case of weak preference, for all $s \in \mathbb{T}$,*

$$\Big(\frac{Z[s, s+\varphi^*_{\kappa t}] - \varphi^*_{\kappa t}}{\sqrt{\kappa}} \colon t \geqslant 0\Big) \Rightarrow (W_t \colon t \geqslant 0)\,,$$

*in distribution on $\mathcal{L}[0,\infty)$, where $(W_t \colon t \geqslant 0)$ is a standard Brownian motion and $(\varphi^*_t)_{t\geqslant 0}$ is the inverse of $(\varphi_t)_{t\geqslant 0}$ given by*

$$\varphi_t = \int_0^{\Phi^{-1}(t)} \frac{1}{f(u)^2}\, du.$$

Our *main* result describes the behaviour of the vertex of *maximal* degree, and reveals an interesting dichotomy between weak and strong forms of preferential attachment.

**Theorem 1.7** (Vertex of maximal degree)**.** *Suppose $f$ is concave. Then we have the following dichotomy:*

**Strong preference.** *If*

$$\sum_{k=0}^{\infty} \frac{1}{f(k)^2} < \infty,$$

*then with probability one there exists a persistent hub, i.e. there is a single vertex which has maximal indegree for all but finitely many times.*

**Weak preference.** *If*

$$\sum_{k=0}^{\infty} \frac{1}{f(k)^2} = \infty,$$

*then with probability one there exists no persistent hub and the time of birth, or index, of the current hub tends to infinity in probability.*

**Remark 1.8.** Without the assumption of concavity of $f$, the assertion remains true in the weak preference regime. In the strong preference regime our results still imply that, almost surely, the number of vertices, which at some time have maximal indegree, is finite.

**Remark 1.9.** In the weak preference case the information about the order of the vertices is asymptotically lost: as a consequence of the proof of Theorem 1.7, we have for two nodes $s < s'$ in $\mathbb{T}$ that

$$\lim_{t\to\infty} \mathbb{P}(Z[s,t] > Z[s',t]) = \tfrac{1}{2},$$

a phenomenon reminiscent of *propagation of chaos*. Conversely, in the strong preference case, the information about the order is not lost completely and one has

$$\lim_{t\to\infty} \mathbb{P}(Z[s,t] > Z[s',t]) > \tfrac{1}{2}.$$

Investigations so far were centred around *typical* vertices in the network. Large deviation principles, as provided below, are the main tool to analyse *exceptional* vertices in the random network. Throughout we use the large-deviation terminology of Dembo and Zeitouni (1998) and, from this point on, the focus is on the weak preference case.

Our aim is to determine the typical age and indegree evolution of the hub. For this purpose we assume that

- $f$ is regularly varying with index $0 \leqslant \alpha < \frac{1}{2}$,
- for some $\eta < 1$, we have $f(j) \leqslant \eta(j+1)$ for all $j \in \mathbb{Z}_+$. (1)

We set $\bar{f} := f \circ \Phi^{-1}$, and recall from Lemma A.1 in the appendix that we can represent $\bar{f}$ as $\bar{f}(u) = u^{\alpha/(1-\alpha)}\, \bar{\ell}(u)$ for $u > 0$, where $\bar{\ell}$ is a slowly varying function. We denote by $\mathcal{I}[0,\infty)$ the space of nondecreasing functions $x\colon [0,\infty) \to \mathbb{R}$ with $x(0) = 0$ endowed with the topology of uniform convergence on compact subintervals of $[0,\infty)$.

**Theorem 1.10** (Large deviation principles)**.** *Under assumption* (1)*, for every* $s \in \mathbb{T}$*, the family of functions*

$$\Big(\frac{1}{\kappa} Z[s, s+\kappa t] \colon t \geqslant 0\Big)_{\kappa>0}$$

*satisfies large deviation principles on the space* $\mathcal{I}[0,\infty)$*,*

- *with speed* $(\kappa^{\frac{1}{1-\alpha}} \bar{\ell}(\kappa))$ *and good rate function*

$$J(x) = \begin{cases} \int_0^\infty x_t^{\frac{\alpha}{1-\alpha}} [1 - \dot{x}_t + \dot{x}_t \log \dot{x}_t]\, dt & \text{if } x \text{ is absolutely continuous,} \\ \infty & \text{otherwise.} \end{cases}$$

- *and with speed* $(\kappa)$ *and good rate function*

$$K(x) = \begin{cases} a\, f(0) & \text{if } x_t = (t-a)_+ \text{ for some } a \geqslant 0, \\ \infty & \text{otherwise.} \end{cases}$$

**Remark 1.11.** The large deviation principle states, in particular, that the most likely deviation from the growth behaviour in the law of large numbers is having zero indegree for some (unusually long) time, and after that time typical behaviour kicking in.

More important for our purpose is a moderate deviation principle, which describes deviations on a finer scale. Similar as before, we denote by $\mathcal{L}(0,\infty)$ the space of càdlàg functions $x \colon (0,\infty) \to \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets of $(0,\infty)$, and always use the convention $x_0 := \liminf_{t\downarrow 0} x_t$.

**Theorem 1.12** (Moderate deviation principle)**.** *Suppose* (1) *and that* $(a_\kappa)$ *is regularly varying, so that the limit*

$$c := \lim_{\kappa\uparrow\infty} a_\kappa\, \kappa^{\frac{2\alpha-1}{1-\alpha}}\, \bar{\ell}(\kappa) \in [0,\infty)$$

*exists. If* $\kappa^{\frac{1-2\alpha}{2-2\alpha}}\, \bar{\ell}(\kappa)^{-\frac{1}{2}} \ll a_\kappa \ll \kappa$*, then, for any* $s \in \mathbb{T}$*, the family of functions*

$$\Big(\frac{Z[s, s+\kappa t] - \kappa t}{a_\kappa} \colon t \geqslant 0\Big)_{\kappa>0}$$

*satisfies a large deviation principle on* $\mathcal{L}(0,\infty)$ *with speed* $(a_\kappa^2\, \kappa^{\frac{2\alpha-1}{1-\alpha}}\, \bar{\ell}(\kappa))$ *and good rate function*

$$I(x) = \begin{cases} \frac{1}{2}\int_0^\infty (\dot{x}_t)^2\, t^{\frac{\alpha}{1-\alpha}}\, dt - \frac{1}{c}\, f(0)\, x_0 & \text{if } x \text{ is absolutely continuous and } x_0 \leqslant 0, \\ \infty & \text{otherwise,} \end{cases}$$

*where we use the convention* $1/0 = \infty$*.*

**Remark 1.13.** If $c = \infty$ there is still a moderate deviation principle on the space of functions $x \colon (0,\infty) \to \mathbb{R}$ with the topology of pointwise convergence. However, the rate function $I$, which has the same form as above with $1/\infty$ interpreted as zero, fails to be a good rate function.

**Remark 1.14.** Under assumption (1) the central limit theorem of Proposition 1.6 can be stated as a complement to the moderate deviation principle: For $a_\kappa \sim \kappa^{\frac{1-2\alpha}{2-2\alpha}} \bar{\ell}(\kappa)^{-\frac{1}{2}}$, we have

$$\Big(\frac{Z[s, s+\kappa t] - \kappa t}{a_\kappa} \colon t \geqslant 0\Big) \Rightarrow \Big(\sqrt{\tfrac{1-\alpha}{1-2\alpha}}\, W_{t^{\frac{1-2\alpha}{1-\alpha}}} \colon t \geqslant 0\Big).$$

See Section 2.1 for details.

Our final result describes weak limit laws for index and degree of the vertex of maximal degree. This result relies on the moderate deviation principle above.

**Theorem 1.15** (Limit law for age and degree of the vertex of maximal degree)**.** *Suppose $f$ is regularly varying with index $\alpha < \frac{1}{2}$. Define $s_t^*$ to be the index of the hub at time $t$, and $Z_t^{\max} = Z[s_t^*, t]$ to be the corresponding maximal indegree. One has, in probability,*

$$s_t^* \sim Z_t^{\max} - t \sim \frac{1}{2}\frac{1-\alpha}{1-2\alpha}\frac{t^{\frac{1-2\alpha}{1-\alpha}}}{\bar{\ell}(t)} = \frac{1}{2}\frac{1-\alpha}{1-2\alpha}\frac{t}{\bar{f}(t)}.$$

*Moreover, in probability on $\mathcal{L}(0,\infty)$,*

$$\lim_{t\to\infty}\Big(\frac{Z[s_t^*, s_t^* + tu] - tu}{t^{\frac{1-2\alpha}{1-\alpha}}\bar{\ell}(t)^{-1}} : u \geqslant 0\Big) = \Big(\tfrac{1-\alpha}{1-2\alpha}\big(u^{\frac{1-2\alpha}{1-\alpha}} \wedge 1\big) : u \geqslant 0\Big).$$

**Remark 1.16.** In terms of the natural scaling, we get for the index $m_n^*$ of the hub and the maximal indegree $\mathcal{Z}_n^{\max}$ at natural time $n \in \mathbb{N}$ that, in probability,

$$\log m_n^* \sim \frac{1}{2}\frac{1-\alpha}{1-2\alpha}\frac{(\log n)^{\frac{1-2\alpha}{1-\alpha}}}{\bar{\ell}(\log n)}$$

and

$$\mathcal{Z}_n^{\max} - \Phi^{-1}(\Psi(n)) \sim \frac{1}{2}\frac{1-\alpha}{1-2\alpha}\log n.$$

The remainder of this paper is devoted to the proofs of these results. Rather than proving the results in the order in which they are stated, we proceed by the techniques used. Section 2 is devoted to martingale techniques, which in particular prove the law of large numbers, Proposition 1.4, and the central limit theorem, Proposition 1.6. We also prove a property of the martingale limit which is crucial in the proof of Theorem 1.7. Section 3 is using Markov chain techniques and provides the proof of Theorem 1.1. In Section 4 we collect the large deviation techniques, proving Theorem 1.10 and Theorem 1.12. Section 5 combines the various techniques to prove our main result, Theorem 1.7, along with Theorem 1.15. An appendix collects the auxiliary statements from the theory of regular variation.

# 2 Martingale techniques

In this section we identify a martingale associated with the degree evolution of a vertex, and study its properties. This will be a vital tool in the further analysis of the network.

## 2.1 Martingale convergence

**Lemma 2.1.** *Fix $s \in \mathbb{T}$ and represent $Z[s,\cdot\,]$ as*

$$Z[s,t] = t - s + M_t.$$

*Then $(M_t)_{t\in\mathbb{T},t\geqslant s}$ is a martingale. Moreover, the martingale converges if and only if*

$$\int_0^\infty \frac{1}{f(u)^2}\,du < \infty, \tag{2}$$

*and otherwise it satisfies the following functional central limit theorem: Let*

$$\varphi_t := \int_0^{\Phi^{-1}(t)} \frac{1}{f(v)^2}\,dv = \int_0^t \frac{1}{\bar f(v)}\,dv,$$

*and denote by $\varphi^*\colon [0,\infty)\to[0,\infty)$ the inverse of $(\varphi_t)$; then the martingales*

$$M^\kappa := \Big(\frac{1}{\sqrt{\kappa}} M_{s+\varphi^*_{\kappa t}}\Big)_{t\geqslant 0} \qquad \text{for } \kappa>0$$

*converge in distribution to standard Brownian motion as $\kappa$ tends to infinity. In any case the processes $(\frac{1}{\kappa}Z[s,s+\kappa t])_{t\geqslant 0}$ converge, as $\kappa\uparrow\infty$, almost surely, in $\mathcal{L}[0,\infty)$ to the identity.*

**Proof.** For $t=\Psi(n)\in\mathbb{T}$ we denote by $\Delta t$ the distance between $t$ and its right neighbour in $\mathbb{T}$, i.e.

$$\Delta t = \frac{1}{n} = \frac{1}{\Psi^{-1}(t)}. \tag{3}$$

One has

$$\begin{aligned}\mathbb{E}\big[Z[s,t+\Delta t]-Z[s,t]\,\big|\,\mathcal{G}_n\big] &= \mathbb{E}\big[\Phi\circ\mathcal{Z}[i,n+1]-\Phi\circ\mathcal{Z}[i,n]\,\big|\,\mathcal{G}_n\big]\\ &= \frac{f(\mathcal{Z}[i,n])}{n}\times\frac{1}{f(\mathcal{Z}[i,n])} = \frac{1}{n}.\end{aligned}$$

Moreover,

$$\langle M\rangle_{t+\Delta t}-\langle M\rangle_t = \Big(1-\bar f(Z[s,t])\Delta t\Big)\frac{1}{\bar f(Z[s,t])}\Delta t \leqslant \frac{1}{\bar f(Z[s,t])}\Delta t. \tag{4}$$

Observe that by Doob's martingale inequality and the uniform boundedness of $\bar f(\cdot)^{-1}$ one has

$$a_i := \frac{\mathbb{E}[\sup_{s\leqslant t\leqslant s+2^{i+1}}|M_t|^2]}{\big(2^{i/2}\log 2^i\big)^2} \leqslant C\,\frac{1}{i^2},$$

where $C=C(f(0))$ is a constant only depending on $f(0)$. Moreover, by Chebyshev, one has

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\geqslant s+2^i}\frac{|M_t|}{\sqrt{t-s}\log(t-s)}\geqslant 1\Big) &\leqslant \sum_{k=i}^\infty \mathbb{P}\Big(\sup_{s+2^k\leqslant t\leqslant s+2^{k+1}}\frac{M_t^2}{(t-s)\log^2(t-s)}\geqslant 1\Big)\\ &\leqslant \sum_{k=i}^\infty \mathbb{E}\Big[\sup_{s+2^k\leqslant t\leqslant s+2^{k+1}}\frac{M_t^2}{(t-s)\log^2(t-s)}\Big] \leqslant \sum_{k=i}^\infty a_k < \infty.\end{aligned}$$

Letting $i$ tend to infinity, we conclude that almost surely

$$\limsup_{t\to\infty}\frac{|M_t|}{\sqrt{t-s}\log(t-s)} \leqslant 1. \tag{5}$$

In particular, we obtain almost sure convergence of $(\frac{1}{\kappa}Z[s,s+\kappa t])_{t\geqslant 0}$ to the identity. As a consequence of (4), for any $\varepsilon>0$, there exists a random almost surely finite constant $\eta=\eta(\omega,\varepsilon)$ such that, for all $t\geqslant s$,

$$\langle M\rangle_t \leqslant \int_0^{t-s} \frac{1}{f(\Phi^{-1}((1-\varepsilon)u))}\,du+\eta.$$

Note that $\Phi:[0,\infty)\to[0,\infty)$ is bijective and substituting $(1-\varepsilon)\kappa u$ by $\Phi(v)$ leads to

$$\langle M\rangle_t \leqslant \frac{1}{1-\varepsilon}\int_0^{\Phi^{-1}((1-\varepsilon)(t-s))}\frac{1}{f(v)^2}\,dv+\eta \leqslant \frac{1}{1-\varepsilon}\int_0^{\Phi^{-1}(t-s)}\frac{1}{f(v)^2}\,dv+\eta.$$

Thus, condition (2) implies convergence of the martingale $(M_t)$.
We now assume that $(\varphi_t)_{t\geqslant 0}$ converges to infinity. Since $\varepsilon>0$ was arbitrary the above estimate implies that

$$\limsup_{t\to\infty}\frac{\langle M\rangle_t}{\varphi_{t-s}}\leqslant 1, \text{ almost surely.}$$

To conclude the converse estimate note that $\sum_{t\in\mathbb{T}}(\Delta t)^2<\infty$ so that we get with (4) and (5) that

$$\langle M\rangle_t \geqslant \int_0^{t-s}\frac{1}{f(\Phi^{-1}((1+\varepsilon)u))}\,du-\eta \geqslant \frac{1}{1+\varepsilon}\int_0^{\Phi^{-1}(t-s)}\frac{1}{f(v)^2}\,dv-\eta,$$

for an appropriate finite random variable $\eta$. Therefore,

$$\lim_{t\to\infty}\frac{\langle M\rangle_t}{\varphi_{t-s}}=1 \qquad \text{almost surely.} \tag{6}$$

The jumps of $M^\kappa$ are uniformly bounded by a deterministic value that tends to zero as $\kappa$ tends to $\infty$. By a functional central limit theorem for martingales (see, e.g., Theorem 3.11 in Jacod and Shiryaev (2003)), the central limit theorem follows once we establish that, for any $t\geqslant 0$,

$$\lim_{\kappa\to\infty}\langle M^\kappa\rangle_t=t \quad \text{in probability,}$$

which is an immediate consequence of (6). □

**Proof of Remark 1.14.** We suppose that $f$ is regularly varying with index $\alpha<\frac{1}{2}$. By the central limit theorem the processes

$$(Y_t^\kappa:t\geqslant 0):=\Big(\frac{Z[s,s+\varphi^*_{t\varphi_\kappa}]-\varphi^*_{t\varphi_\kappa}}{\sqrt{\varphi(\kappa)}}:t\geqslant 0\Big) \qquad \text{for } \kappa>0$$

converge in distribution to the Wiener process $(W_t)$ as $\kappa$ tends to infinity. For each $\kappa>0$ we consider the time change $(\tau_t^\kappa)_{t\geqslant 0}:=(\varphi_{\kappa t}/\varphi_\kappa)$. Using that $\varphi$ is regularly varying with parameter $\frac{1-2\alpha}{1-\alpha}$ we find uniform convergence on compacts:

$$(\tau_t^\kappa)\to(t^{\frac{1-2\alpha}{1-\alpha}})=:(\tau_t) \qquad \text{as } \kappa\to\infty.$$

Therefore,

$$\Big(\frac{Z[s,s+\kappa t]-\kappa t}{\sqrt{\varphi(\kappa)}}:t\geqslant 0\Big)=(Y^\kappa_{\tau_t^\kappa}:t\geqslant 0)\Rightarrow(W_{\tau_t}:t\geqslant 0).$$

and, as shown in Lemma A.1, $\varphi_\kappa\sim\frac{1-\alpha}{1-2\alpha}\kappa^{\frac{1-2\alpha}{1-\alpha}}\bar\ell(\kappa)^{-1}$. □

## 2.2 Absolute continuity of the law of $M_\infty$

In the sequel, we consider the martingale $(M_t)_{t\geqslant s, t\in\mathbb{T}}$ given by $Z[s,t]-(t-s)$ for a fixed $s\in\mathbb{T}$ in the case of *strong* preference. We denote by $M_\infty$ the limit of the martingale.

**Proposition 2.2.** *If $f$ is concave, then the distribution of $M_\infty$ is absolutely continuous with respect to Lebesgue measure.*

**Proof.** For ease of notation, we denote $Y_t = Z[s,t]$, for $t\in\mathbb{T}$, $t\geqslant s$. Moreover, we fix $c>0$ and let $A_t$ denote the event that $Y_u\in[u-c,u+c]$ for all $u\in[s,t]\cap\mathbb{T}$.
Now observe that for two neighbours $v_-$ and $v$ in $\mathbb{S}$

$$\mathbb{P}(Y_{t+\Delta t}=v;A_t) = (1-\bar f(v)\,\Delta t)\,\mathbb{P}(Y_t=v;A_t) + \bar f(v_-)\,\Delta t\,\mathbb{P}(Y_t=v_-;A_t). \tag{7}$$

Again we use the notation $\Delta t = \frac{1}{\Psi^{-1}(t)}$. Moreover, we denote $\Delta\bar f(v) = \bar f(v)-\bar f(v_-)$. In the first step of the proof we derive an upper bound for

$$h(t) = \max_{v\in\mathbb{S}} \mathbb{P}(Y_t=v;A_t) \qquad \text{for } t\in\mathbb{T}, t\geqslant s.$$

With (7) we conclude that

$$\mathbb{P}(Y_{t+\Delta t}=v;A_t) \leqslant (1-\Delta\bar f(v)\,\Delta t)\,h(t).$$

For $w\geqslant 0$ we denote $\varsigma(w)=\max\mathbb{S}\cap[0,w]$. Due to the concavity of $f$, we get that

$$h(t+\Delta t) \leqslant (1-\Delta\bar f(\varsigma(t+c+1))\,\Delta t)\,h(t).$$

Consequently,

$$h(t) \leqslant \prod_{u\in[s,t)\cap\mathbb{T}} (1-\Delta\bar f(\varsigma(u+c+1))\,\Delta u)$$

and using that $\log(1+x)\leqslant x$ we obtain

$$h(t) \leqslant \exp\Bigl(-\sum_{u\in[s,t)\cap\mathbb{T}} \Delta\bar f(\varsigma(u+c+1))\,\Delta u\Bigr). \tag{8}$$

We continue with estimating the sum $\Sigma$ in the latter exponential:

$$\Sigma = \sum_{u\in[s,t)\cap\mathbb{T}} \Delta\bar f(\varsigma(u+c+1))\,\Delta u \geqslant \int_s^t \Delta\bar f(\varsigma(u+c+1))\,du.$$

Next, we denote by $f^{\mathrm{lin}}$ the continuous piecewise linear interpolation of $f|_{\mathbb{N}_0}$. Analogously, we set $\Phi^{\mathrm{lin}}(v) = \int_0^v \frac{1}{f^{\mathrm{lin}}(u)}\,du$ and $\bar f^{\mathrm{lin}}(v) = f^{\mathrm{lin}}\circ(\Phi^{\mathrm{lin}})^{-1}(v)$. Using again the concavity of $f$ we conclude that

$$\int_s^t \Delta\bar f(\varsigma(u+c+1))\,du \geqslant \int_s^t (f^{\mathrm{lin}})'(\Phi^{-1}(u+c+1))\,du,$$

and that

$$f^{\mathrm{lin}}\geqslant f \;\Rightarrow\; \Phi^{\mathrm{lin}}\leqslant\Phi \;\Rightarrow\; (\Phi^{\mathrm{lin}})^{-1}\geqslant\Phi^{-1} \;\Rightarrow\; (f^{\mathrm{lin}})'\circ(\Phi^{\mathrm{lin}})^{-1}\leqslant(f^{\mathrm{lin}})'\circ\Phi^{-1}.$$

Hence,

$$\Sigma \geqslant \int_s^t (f^{\rm lin})'(\Phi^{-1}(u+c+1))\,du \geqslant \int_s^t (f^{\rm lin})' \circ (\Phi^{\rm lin})^{-1}(u+c+1)\,du.$$

For Lebesgue almost all arguments one has

$$(\bar f^{\rm lin})' = (f^{\rm lin} \circ (\Phi^{\rm lin})^{-1})' = (f^{\rm lin})' \circ (\Phi^{\rm lin})^{-1} \cdot ((\Phi^{\rm lin})^{-1})' = (f^{\rm lin})' \circ (\Phi^{\rm lin})^{-1} \cdot f^{\rm lin} \circ (\Phi^{\rm lin})^{-1}$$

so that

$$(f^{\rm lin})' \circ (\Phi^{\rm lin})^{-1} = \frac{(\bar f^{\rm lin})'}{\bar f^{\rm lin}} = (\log \bar f^{\rm lin})'.$$

Consequently,

$$\Sigma \geqslant \log \bar f^{\rm lin}(t+c+1) - \log \bar f^{\rm lin}(s+c+1)$$

Using that $f^{\rm lin} \geqslant f$ and $(\Phi^{\rm lin})^{-1} \geqslant \Phi^{-1}$ we finally get that

$$\Sigma \geqslant \log \bar f(t+c+1) - \log c^*,$$

where $c^*$ is a positive constant not depending on $t$. Plugging this estimate into (7) we get

$$h(t) \leqslant \frac{c^*}{\bar f(t+c+1)}.$$

Fix now an interval $I \subset \mathbb{R}$ of finite length and note that

$$\mathbb{P}(M_t \in I; A_t) = \mathbb{P}(Y_t \in t-s+I; A_t) \leqslant \#[(t-s+I) \cap \mathbb{S} \cap A_t] \cdot h(t).$$

Now $(t-s+I) \cap \mathbb{S} \cap A_t$ is a subset of $[t-c, t+c]$ and the minimal distance of two distinct elements is bigger than $\frac{1}{\bar f(t+c)}$. Therefore, $\#[(t-s+I) \cap \mathbb{S} \cap A_t] \leqslant |I|\, \bar f(t+c) + 1$, and

$$\mathbb{P}(M_t \in I; A_t) \leqslant c^*\,|I| + \frac{c^*}{\bar f(t+c)}.$$

Moreover, for any open and thus immediately also for any arbitrary interval $I$ one has

$$\mathbb{P}(M_\infty \in I; A_\infty) \leqslant \liminf_{t\to\infty} \mathbb{P}(M_t \in I; A_t) \leqslant c^*\,|I|,$$

where $A_\infty = \bigcap_{t \in [s,\infty)\cap \mathbb{T}} A_t$. Consequently, the Borel measure $\mu_c$ on $\mathbb{R}$ given by $\mu_c(E) = \mathbb{E}[\mathbb{1}_{A_\infty} \cdot \mathbb{1}_E(M_\infty)]$, is absolutely continuous with respect to Lebesgue measure. The distribution $\mu$ of $M_\infty$, i.e. $\mu(E) = \mathbb{E}[\mathbb{1}_E(M_\infty)]$, can be written as monotone limit of the absolutely continuous measures $\mu_c$ ($c \in \mathbb{N}$), and it is thus also absolutely continuous. □

# 3 The empirical indegree distribution

In this section we prove Theorem 1.1. For $k \in \mathbb{Z}_+$ and $n \in \mathbb{N}$ let $\mu_k(n) = \mathbb{E}[X_k(n)]$ and $\mu(n) = (\mu_k(n))_{k\in\mathbb{Z}_+}$. We first show that $(\mu(n))_{n\in\mathbb{N}}$ converges to $\mu = (\mu_k)_{k\in\mathbb{Z}_+}$ as $n$ tends to infinity. We start by deriving a recursive representation for $\mu(n)$. For $k \in \mathbb{Z}_+$,

$$\begin{aligned}\mathbb{E}[X_k(n+1)|X(n)] &= \frac{1}{n+1}\Big(\sum_{i=1}^n \mathbb{E}\Big[-\mathbb{1}_{\{\mathcal{Z}[i,n]=k<\mathcal{Z}[i,n+1]\}} + \mathbb{1}_{\{\mathcal{Z}[i,n]<k=\mathcal{Z}[i,n+1]\}}\Big| X(n)\Big] \\ &\qquad\qquad + nX_k(n) + \mathbb{1}_{\{k=0\}}\Big) \\ &= X_k(n) + \frac{1}{n+1}\Big[-nX_k(n)\frac{f(k)}{n} + nX_{k-1}(n)\frac{f(k-1)}{n} - X_k(n) + \mathbb{1}_{\{k=0\}}\Big].\end{aligned}$$

Thus the linearity and the tower property of conditional expectation gives

$$\mu_k(n+1) = \mu_k(n) + \frac{1}{n+1}\big(f(k-1)\mu_{k-1}(n) - (1+f(k))\mu_k(t) + \mathbb{1}_{\{k=0\}}\big).$$

Now defining $Q \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ as

$$Q = \begin{pmatrix} -f(0) & f(0) & & & \\ 1 & -(f(1)+1) & f(1) & & \\ 1 & & -(f(2)+1) & f(2) & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{9}$$

and conceiving $\mu(n)$ as a row vector we can rewrite the recursive equation as

$$\mu(n+1) = \mu(n)\Big(I + \frac{1}{n+1}\,Q\Big),$$

where $I = (\delta_{i,j})_{i,j\in\mathbb{N}}$ denotes the unit matrix. Next we show that $\mu$ is a probability distribution with $\mu Q = 0$. By induction, we get that

$$1 - \sum_{l=0}^{k} \mu_l = \prod_{l=0}^{k} \frac{f(l)}{1+f(l)}$$

for any $k \in \mathbb{Z}_+$. Since $\sum_{l=0^\infty} 1/f(l) \geqslant \sum_{l=0}^{\infty} 1/(l+1) = \infty$ it follows that $\mu$ is a probability measure on $\mathbb{Z}_+$. Moreover, it is straight-forward to verify that

$$f(0)\mu_0 = 1 - \mu_0 = \sum_{l=1}^{\infty} \mu_l$$

and that for all $k \in \mathbb{Z}_+$

$$f(k-1)\,\mu_{k-1} = (1+f(k))\,\mu_k,$$

hence $\mu Q = 0$.

Now we use the matrices $P^{(n)} := I + \frac{1}{n+1}\,Q$ to define an inhomogeneous Markov process. The entries of each row of $P^{(n)}$ sum up to 1 but (as long as $f$ is not bounded) each $P^{(n)}$ contains negative entries. Nonetheless one can use the $P^{(n)}$ as a time inhomogeneous Markov kernel as long as at the starting time $m \in \mathbb{N}$ the starting state $l \in \mathbb{Z}_+$ satisfies $l \leqslant m-1$.

We denote for any admissible pair $l, m$ by $(Y_n^{l,m})_{n\,\geqslant\, m}$ a Markov chain starting at time $m$ in state $l$ having transition kernels $(P^{(n)})_{n\,\geqslant\, m}$. Due to the recursive equation we now have

$$\mu_k(n) = \mathbb{P}(Y_n^{0,1} = k).$$

Next, fix $k \in \mathbb{Z}_+$, let $m > k$ arbitrary, and denote by $\nu$ the restriction of $\mu$ to the set $\{m, m+1, \dots\}$. Since $\mu$ is invariant under each $P^{(n)}$ we get

$$\mu_k = (\mu P^{(m)} \dots P^{(n)})_k = \sum_{l=0}^{m-1} \mu_l\, \mathbb{P}(Y_n^{l,m} = k) + (\nu P^{(m)} \dots P^{(n)})_k.$$

Note that in the $n$-th step of the Markov chain, the probability to jump to state zero is $\frac{1}{n+1}$ for all original states in $\{1, \dots, n-1\}$ and bigger than $\frac{1}{n+1}$ for the original state 0. Thus one can couple the Markov chains $(Y_n^{l,m})$ and $(Y_n^{0,1})$ in such a way that

$$\mathbb{P}(Y_{n+1}^{l,m} = Y_{n+1}^{0,1} = 0 \,|\, Y_n^{l,m} \neq Y_n^{0,1}) = \tfrac{1}{n+1},$$

and that once the processes meet at one site they stay together. Then

$$\mathbb{P}(Y_n^{l,m} = Y_n^{0,1}) \geqslant 1 - \prod_{i=m}^{n-1} \frac{i}{i+1} \longrightarrow 1.$$

Thus $(\nu P^{(m)} \dots P^{(n)})_k \in [0, \mu([m,\infty))]$ implies that

$$\limsup_{n\to\infty} \Big| \mu_k - \mathbb{P}(Y_n^{(0,1)} = k) \sum_{l=0}^{m-1} \mu_l^* \Big| \leqslant \mu([m,\infty)).$$

As $m \to \infty$ we thus get that

$$\lim_{n\to\infty} \mu_k(n) = \mu_k.$$

In the next step we show that the sequence of the empirical indegree distributions $(X(n))_{n\in\mathbb{N}}$ converges almost surely to $\mu$. Note that $n\,X_k(n)$ is a sum of $n$ independent Bernoulli random variables. Thus Chernoff's inequality (Chernoff (1981)) implies that for any $t > 0$

$$\mathbb{P}\big(n\,X_k(n) \leqslant n\,(\mathbb{E}[X_k(n)] - t)\big) \leqslant e^{-nt^2/(2\mathbb{E}[X_k(n)])} = e^{-nt^2/(2\mu_k(n))}.$$

Since

$$\sum_{n=1}^{\infty} e^{-nt^2/(2\mu_k(n))} < \infty,$$

Borel-Cantelli implies that almost surely $\liminf_{n\to\infty} X_k(n) \geqslant \mu_k$ for all $k \in \mathbb{Z}_+$. This establishes almost sure convergence of $(X(n))$ to $\mu$.

We still need to show that the conditional law of the outdegree of a new node converges almost surely in the weak topology to a Poisson distribution. In the first step we will prove that, for $\eta \in (0,1)$, and the affine linear attachment rule $f(k) = \eta k + 1$, one has almost sure convergence of $Y_n := \frac{1}{n}\sum_{m=1}^n \mathcal{Z}[m,n] = \langle X(n), \mathrm{id}\rangle$ to $y := 1/(1-\eta)$. First observe that

$$Y_{n+1} = \tfrac{1}{n+1}\big[nY_n + \sum_{m=1}^{n} \Delta\mathcal{Z}[m,n]\big] = Y_n + \tfrac{1}{n+1}\big[-Y_n + \sum_{m=1}^{n} \Delta\mathcal{Z}[m,n]\big],$$

where $\Delta\mathcal{Z}[m,n] := \mathcal{Z}[m,n+1] - \mathcal{Z}[m,n]$. Given the past $\mathcal{F}_n$ of the network formation, each $\Delta\mathcal{Z}[m,n]$ is independent Bernoulli distributed with success probability $\frac{1}{n}(\eta\mathcal{Z}[m,n]+1)$. Consequently,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= Y_n + \tfrac{1}{n+1}\big[-Y_n + \sum_{m=1}^{n} \tfrac{1}{n}(\eta\mathcal{Z}[m,n]+1)\big] \\ &= Y_n + \tfrac{1}{n+1}\big[-(1-\eta)Y_n + 1\big],\end{aligned}$$

and

$$\langle Y\rangle_{n+1}-\langle Y\rangle_n\leqslant\tfrac{1}{(n+1)^2}\sum_{m=1}^{n}\tfrac{1}{n}(\eta\mathcal{Z}[m,n]+1)=\tfrac{1}{(n+1)^2}[\eta Y_n+1]. \tag{10}$$

Now note that due to Theorem 1.7 (which can be used here, as it will be proved independently of this section) there is a single node that has maximal indegree for all but finitely many times. Let $m^*$ denote the random node with this property. With Remark 1.5 we conclude that almost surely

$$\log\mathcal{Z}[m^*,n]\sim\log n^{\eta}. \tag{11}$$

Since for sufficiently large $n$

$$Y_n=\frac{1}{n}\sum_{m=1}^{n}\mathcal{Z}[m,n]\leqslant\mathcal{Z}[m^*,n],$$

equations (10) and (11) imply that $\langle Y\rangle_\cdot$ converges almost surely to a finite random variable. Next, represent the increment $Y_{n+1}-Y_n$ as

$$Y_{n+1}-Y_n=\tfrac{1}{n+1}\big[-(1-\eta)Y_n+1\big]+\Delta M_{n+1}, \tag{12}$$

where $\Delta M_{n+1}$ denotes a martingale difference. We shall denote by $(M_n)_{n\in\mathbb{N}}$ the corresponding martingale, that is $M_n=\sum_{m=2}^{n}\Delta M_m$. Since $\langle Y\rangle_\cdot$ is convergent, the martingale $(M_n)$ converges almost surely. Next, we represent (12) in terms of $\bar{Y}_n=Y_n-y$ as the following inhomogeneous linear difference equation of first order:

$$\bar{Y}_{n+1}=\Big(1-\tfrac{1-\eta}{n+1}\Big)\bar{Y}_n+\Delta M_{n+1}.$$

The corresponding starting value is $\bar{Y}_1=Y_1-y=-y$, and we can represent its solution as

$$\bar{Y}_n=-y\,h_n^1+\sum_{m=2}^{n}\Delta M_m\,h_n^m$$

for

$$h_n^m:=\begin{cases}0 & \text{if } n<m\\ \prod_{l=m+1}^{n}\Big(1-\frac{1-\eta}{l}\Big) & \text{if } n\geqslant m.\end{cases}$$

Setting $\Delta h_n^m=h_n^m-h_n^{m-1}$ we conclude with an integration by parts argument that

$$\begin{aligned}\sum_{m=2}^{n}\Delta M_m\,h_n^m&=\sum_{m=2}^{n}\Delta M_m\Big(1-\sum_{k=m+1}^{n}\Delta h_n^k\Big)=M_n-\sum_{m=2}^{n}\sum_{k=m+1}^{n}\Delta M_m\,\Delta h_n^k\\&=M_n-\sum_{k=3}^{n}\sum_{m=2}^{k-1}\Delta M_m\,\Delta h_n^k=M_n-\sum_{k=3}^{n}M_{k-1}\,\Delta h_n^k.\end{aligned} \tag{13}$$

Note that $h_n^m$ and $\Delta h_n^m$ tend to 0 as $n$ tends to infinity so that $\sum_{k=m+1}^n \Delta h_n^k = 1 - h_n^m$ tends to 1. With $M_\infty := \lim_{n\to\infty} M_n$ and $\varepsilon_m = \sup_{n\geqslant m} |M_n - M_\infty|$ we derive for $m \leqslant n$

$$\begin{aligned}
&\big|M_n - \sum_{k=3}^{m} M_{k-1}\,\Delta h_n^k - \sum_{k=m+1}^{n} M_{k-1}\,\Delta h_n^k\big| \\
&\quad\leqslant \underbrace{|M_n - M_\infty|}_{\to 0} + \underbrace{\sum_{k=3}^{m} |M_{k-1}|\,\Delta h_n^k}_{\to 0} + \underbrace{\big|\sum_{k=m+1}^{n} (M_\infty - M_{k-1})\,\Delta h_n^k\big|}_{\leqslant \varepsilon_m} + \underbrace{\big(1 - \sum_{k=m+1}^{n} \Delta h_n^k\big)|M_\infty|}_{\to 0}.
\end{aligned}$$

Since $\lim_{m\to\infty} \varepsilon_m = 0$, almost surely, we thus conclude with (13) that $\sum_{m=2}^n \Delta M_m\, h_n^m$ tends to 0. Consequently, $\lim_{n\to\infty} Y_n = y$, almost surely. Next, we show that also $\langle \mu, \mathrm{id}\rangle = y$. Recall that $\mu$ is the unique invariant distribution satisfying $\mu\, Q = 0$ (see (9) for the definition of $Q$). This implies that for any $k \in \mathbb{N}$

$$f(k-1)\,\mu_{k-1} - (f(k)+1)\,\mu_k = 0, \quad \text{or equivalently,} \quad \mu_k = f(k-1)\,\mu_{k-1} - f(k)\,\mu_k$$

Thus

$$\langle \mu, \mathrm{id}\rangle = \sum_{k=1}^{\infty} k\,\mu_k = \sum_{k=1}^{\infty} k\,\big[f(k-1)\,\mu_{k-1} - f(k)\,\mu_k\big].$$

One cannot split the sum into two sums since the individual sums are not summable. However, noticing that the individual term $f(k)\mu_k k \approx k^2 \mu_k$ tends to 0, we can rearrange the summands to obtain

$$\langle \mu, \mathrm{id}\rangle = f(0)\,\mu_0 + \sum_{k=1}^{\infty} f(k)\,\mu_k = \langle \mu, f\rangle = \eta\langle \mu, \mathrm{id}\rangle + 1.$$

This implies that $\langle \mu, \mathrm{id}\rangle = y$ and that for any $m \in \mathbb{N}$

$$\langle X(n), \mathbb{1}_{[m,\infty)} \cdot \mathrm{id}\rangle = \langle X(n), \mathrm{id}\rangle - \langle X(n), \mathbb{1}_{[0,m)} \cdot \mathrm{id}\rangle \to \langle \mu, \mathbb{1}_{[m,\infty)} \cdot \mathrm{id}\rangle, \text{ almost surely.}$$

Now, we switch to general attachment rules. We denote by $f$ an arbitrary attachment rule that is dominated by an affine attachment rule $f^{\mathrm{a}}$. The corresponding degree evolutions will be denoted by $(\mathcal{Z}[m,n])$ and $(\mathcal{Z}^{\mathrm{a}}[m,n])$, respectively. Moreover, we denote by $\mu$ and $\mu^{\mathrm{a}}$ the limit distributions of the empirical indegree distributions. Since by assumption $f \leqslant f^{\mathrm{a}}$, one can couple both degree evolutions such that $\mathcal{Z}[m,n] \leqslant \mathcal{Z}^{\mathrm{a}}[m,n]$ for all $n \geqslant m \geqslant 0$. Now

$$\langle X(n), f\rangle \leqslant \langle X(n), \mathbb{1}_{[0,m)} \cdot f\rangle + \langle X^{\mathrm{a}}(n), \mathbb{1}_{[m,\infty)} \cdot f^{\mathrm{a}}\rangle$$

so that almost surely

$$\limsup_{n\to\infty} \langle X(n), f\rangle \leqslant \langle \mu, \mathbb{1}_{[0,m)} \cdot f\rangle + \langle \mu^{\mathrm{a}}, \mathbb{1}_{[m,\infty)} \cdot f^{\mathrm{a}}\rangle.$$

Since $m$ can be chosen arbitrarily large we conclude that

$$\limsup_{n\to\infty} \langle X(n), f\rangle \leqslant \langle \mu, f\rangle.$$

The converse estimate is an immediate consequence of Fatou's lemma. Hence,

$$\lim_{n\to\infty} \mathbb{E}\Big[\sum_{m=1}^{n} \Delta\mathcal{Z}[m,n]\Big|\mathcal{F}_n\Big] = \langle \mu, f\rangle.$$

Since, conditional on $\mathcal{F}_n$, $\sum_{m=1}^n \Delta\mathcal{Z}[m,n]$ is a sum of independent Bernoulli variables with success probabilities tending uniformly to 0, we finally get that $\mathcal{L}(\sum_{m=1}^n \Delta\mathcal{Z}[m,n]|\mathcal{F}_n)$ converges in the weak topology to a Poisson distribution with parameter $\langle \mu, f\rangle$.

# 4 Large deviations

In this section we derive tools to analyse rare events in the random network. We provide large and moderate deviation principles for the temporal development of the indegree of a given vertex. This will allow us to describe the indegree evolution of the node with maximal indegree in the case of weak preferential attachment. The large and moderate deviation principles are based on an exponential approximation to the indegree evolution processes, which we first discuss.

## 4.1 Exponentially good approximation

In order to analyze the large deviations of the process $Z[s,\cdot]$ (or $\mathcal{Z}[m,\cdot,])$ we use an approximating process. We first do this on the level of occupation measures. For $s\in\mathbb{T}$ and $0\leqslant u<v$ we define

$$T_s[u,v) = \sup\{t'-t : Z[s,t]\geqslant u, Z[s,t']<v, t,t'\in\mathbb{T}\}$$

to be the time the process $Z[s,\cdot]$ spends in the interval $[u,v)$. Similarly, we denote by $T_s[u]$ the time spent in $u$. Moreover, we denote by $(T[u])_{u\in\mathbb{S}}$ a family of independent random variables with each entry $T[u]$ being $\mathsf{Exp}(f(u))$-distributed, and denote

$$T[u,v) := \sum_{\substack{w\in\mathbb{S}\\ u\leqslant w<v}} T[w] \qquad \text{for all } 0\leqslant u\leqslant v.$$

The following lemma shows that $T[u,v)$ is a good approximation to $T_s[u,v)$ in many cases.

**Lemma 4.1.** *Fix $\eta_1\in(0,1)$, let $s\in\mathbb{T}$ and denote by $\tau$ the entry time into $u$ of the process $Z[s,\cdot]$. One can couple $(T_s[u])_{u\in\mathbb{S}}$ and $(T[u])_{u\in\mathbb{S}}$ such that, almost surely,*

$$\mathbb{1}_{\{\bar f(u)\Delta\tau\leqslant\eta_1\}}|T_s[u]-T[u]|\leqslant(1\vee\eta_2\bar f(u))\Delta\tau,$$

*where $\eta_2$ is a constant only depending on $\eta_1$.*

**Proof.** We fix $t\in\mathbb{T}$ with $\bar f(u)\Delta t\leqslant\eta_1$. Note that it suffices to find an appropriate coupling conditional on the event $\{\tau=t\}$. Let $U$ be a uniform random variable and let $F$ and $\bar F$ denote the (conditional) distribution functions of $T[u]$ and $T_s[u]$, respectively. We couple $T[u]$ and $T_s[u]$ by setting $T[u]=F^{-1}(U)$ and $T_s[u]=\bar F^{-1}(U)$, where $\bar F^{-1}$ denotes the right continuous inverse of $\bar F$. The variables $T[u]$ and $T_s[u]$ satisfy the assertion of the lemma if and only if

$$F\big(v-(1\vee\eta_2\bar f(u))\Delta t\big)\leqslant\bar F(v)\leqslant F\big(v+(1\vee\eta_2\bar f(u))\Delta t\big) \qquad \text{for all } v\geqslant 0. \tag{14}$$

We compute

$$1-\bar F(v)=\prod_{\substack{t\leqslant w,w+\Delta w\leqslant t+v\\ w\in\mathbb{T}}}\big(1-\bar f(u)\Delta w\big)=\exp\sum_{\substack{t\leqslant w,w+\Delta w\leqslant t+v\\ u\in\mathbb{T}}}\log\big(1-\bar f(u)\Delta w\big)$$

Next observe that, from a Taylor expansion, for a suitably large $\eta_2>0$, we have $-\bar f(u)\Delta w-\eta_2\,\bar f(u)^2[\Delta w]^2\leqslant\log\big(1-\bar f(u)\Delta w\big)\leqslant-\bar f(u)\Delta w$, so that

$$1-\bar F(v)\leqslant\exp\Big(-\bar f(u)\sum_{\substack{t\leqslant w,w+\Delta w\leqslant t+v\\ w\in\mathbb{T}}}\Delta w\Big)\leqslant\exp\big(-\bar f(u)(v-\Delta t)\big)=1-F(v-\Delta t).$$

This proves the left inequality in (14). It remains to prove the right inequality. Note that

$$1-\bar F(v)\geqslant\exp\Big(-\sum_{\substack{t\leqslant w,w+\Delta w\leqslant t+v\\ w\in\mathbb{T}}}(\bar f(u)\Delta w+\eta_2\,\bar f(u)^2[\Delta w]^2)\Big)$$

and

$$\sum_{\substack{t\leqslant w,w+\Delta w\leqslant t+v\\ w\in\mathbb{T}}}[\Delta w]^2\leqslant\sum_{m=[\Delta t]^{-1}}^{\infty}\frac{1}{m^2}\leqslant\frac{1}{[\Delta t]^{-1}-1}\leqslant\Delta t.$$

Consequently, $1-\bar F(v)\geqslant\exp\{-\bar f(u)(v+\eta_2\bar f(u)\Delta t)\}=1-F(v+\eta_2\bar f(u)\Delta t)$. □

As a direct consequence of this lemma we obtain an exponential approximation.

**Lemma 4.2.** *Suppose that, for some $\eta<1$ we have $f(j)\leqslant\eta(j+1)$ for all $j\in\mathbb{Z}_+$. If*

$$\sum_{j=0}^{\infty}\frac{f(j)^2}{(j+1)^2}<\infty,$$

*then for each $s\in\mathbb{T}$ one can couple $T_s$ with $T$ such that, for all $\lambda\geqslant0$,*

$$\mathbb{P}\big(\sup_{u\in\mathbb{S}}|T_s[0,u)-T[0,u)|\geqslant\lambda+\sqrt{2K}\big)\leqslant4\,e^{-\frac{\lambda^2}{2K}},$$

*where $K>0$ is a finite constant only depending on $f$.*

**Proof.** Fix $s\in\mathbb{T}$ and denote by $\tau_u$ the first entry time of $Z[s,\cdot]$ into the state $u\in\mathbb{S}$. We couple the random variables $T[u]$ and $T_s[u]$ as in the previous lemma and let, for $v\in\mathbb{S}$,

$$M_v=\sum_{\substack{u\in\mathbb{S}\\ u<v}}\big(T_s[u]-T[u]\big)=T_s[0,v)-T[0,v).$$

Then $(M_v)_{v\in\mathbb{S}}$ is a martingale. Moreover, for each $v=\Phi(j)\in\mathbb{S}$ one has $\tau_v\geqslant\Psi(j+1)$ so that $\Delta\tau_v\leqslant1/(j+1)$. Consequently, using the assumption of the lemma one gets that

$$\Delta\tau_v\,\bar f(v)\leqslant\frac{1}{j+1}f(j)=:c_v\leqslant\eta<1.$$

Thus by Lemma 4.1 there exists a constant $\eta' < \infty$ depending only on $f(0)$ and $\eta$ such that the increments of the martingale $(M_v)$ are bounded by

$$|T_s[v] - T[v]| \leqslant \eta' \, c_v.$$

By assumption we have $K := \sum_{v\in\mathbb{S}} c_v^2 < \infty$ an we conclude with Lemma A.4 that for $\lambda \geqslant 0$,

$$\mathbb{P}\big(\sup_{u\in\mathbb{S}} |T_s[0,u) - T[0,u)| \geqslant \lambda + \sqrt{2K}\big) \leqslant 4\, e^{-\frac{\lambda^2}{2K}}.$$

□

We define $(Z_t)_{t\,\geqslant\,0}$ to be the $\mathbb{S}$-valued process given by

$$Z_t := \max\{v \in \mathbb{S} : T[0,v) \leqslant t\}, \tag{15}$$

and start by observing its connection to the indegree evolution.

**Corollary 4.3.** *In distribution on the Skorokhod space, we have*

$$\lim_{s\uparrow\infty} (Z[s,s+t])_{t\,\geqslant\,0} = (Z_t)_{t\,\geqslant\,0}.$$

**Proof.** Recall that Lemma 4.1 provides a coupling between $(T_s[u])_{u\in\mathbb{S}}$ and $(T[u])_{u\in\mathbb{S}}$ for any fixed $s \in \mathbb{T}$. We may assume that the coupled random variables $(\bar{T}_s[u])_{u\in\mathbb{S}}$ and $(\bar{T}[u])_{u\in\mathbb{S}}$ are defined on the same probability space for all $s \in \mathbb{T}$ (though this does not respect the joint distributions of $(T_s[u])_{u\in\mathbb{S}}$ for different values of $s \in \mathbb{T}$). Denote by $(\bar{Z}[s,\cdot])_{s\in\mathbb{T}}$ and $(\bar{Z}_t)$ the corresponding processes such that $\bar{T}_s[0,u) + s = s + \sum_{v<u} \bar{T}_s[v]$ and $\bar{T}[0,u) = \sum_{v<u} \bar{T}[v]$ are the entry times of $(\bar{Z}[s,s+t])$ and $(\bar{Z}_t)$ into the state $u \in \mathbb{S}$. By Lemma 4.1 one has that, almost surely, $\lim_{s\uparrow\infty} \bar{T}_s[0,u) = \bar{T}[0,u)$ and therefore one obtains almost sure convergence of $(\bar{Z}[s,s+t])_{t\geqslant 0}$ to $(\bar{Z}_t)_{t\geqslant 0}$ in the Skorokhod topology. □

**Proposition 4.4.** *Uniformly in $s$, the processes*

- $(\frac{1}{\kappa} Z_{\kappa t} \colon t \geqslant 0)_{\kappa>0}$ *and* $(\frac{1}{\kappa} Z[s,s+\kappa t] \colon t \geqslant 0)_{\kappa>0}$*;*
- $(\frac{1}{a_\kappa}(Z_{\kappa t} - \kappa t) \colon t \geqslant 0)_{\kappa>0}$ *and* $(\frac{1}{a_\kappa}(Z[s,s+\kappa t] - \kappa t) \colon t \geqslant 0)_{\kappa>0}$*,*

*are exponentially equivalent on the scale of the large, respectively, moderate deviation principles.*

**Proof.** We only present the proof for the first large deviation principle of Theorem 1.10 since all other statements can be inferred analogously.
We let $U_\delta(x)$ denote the open ball around $x \in \mathcal{I}[0,\infty)$ with radius $\delta > 0$ in an arbitrarily fixed metric $d$ generating the topology of uniform convergence on compacts, and, for fixed $\eta > 0$, we cover the compact set $K = \{x \in \mathcal{I}[0,\infty) \colon I(x) \leqslant \eta\}$ with finitely many balls $(U_\delta(x))_{x\in\mathbb{I}}$, where $\mathbb{I} \subset K$. Since every $x \in \mathbb{I}$ is continuous, we can find $\varepsilon > 0$ such that for every $x \in \mathbb{I}$ and increasing and right continuous $\tau : [0,\infty) \to [0,\infty)$ with $|\tau(t) - t| \leqslant \varepsilon$,

$$y \in U_\delta(x) \;\Rightarrow\; y_{\tau(\cdot)} \in U_{2\delta}(x).$$

For fixed $s \in \mathbb{T}$ we couple the occupation times $(T_s[0,u))_{u\in\mathbb{S}}$ and $(T[0,u))_{u\in\mathbb{S}}$ as in Lemma 4.2, and hence implicitly the evolutions $(Z[s,t])_{t\geqslant s}$ and $(Z_t)_{t\geqslant 0}$. Next, note that $Z[s,s+\,\cdot\,]$ can be

transformed into $Z_\cdot$ by applying a time change $\tau$ with $|\tau(t) - t| \leqslant \sup_{u\in\mathbb{S}} |T_s[0, u) - T[0, u)|$. Consequently,

$$\mathbb{P}\Big(d\big(\tfrac{1}{\kappa}Z[s, s + \kappa\,\cdot\,], \tfrac{1}{\kappa}Z_{\kappa\cdot}\big) \geqslant 3\delta\Big) \leqslant \mathbb{P}\Big(\tfrac{1}{\kappa}Z_{\kappa\cdot} \notin \bigcup_{x\in\mathbb{I}} U_\delta(x)\Big) + \mathbb{P}\Big(\sup_{u\in\mathbb{S}} |\bar{T}_s[0, u) - \bar{T}[0, u)| \geqslant \kappa\varepsilon\Big),$$

and an application of Lemma 4.2 gives a uniform upper bound in $s$, namely

$$\limsup_{\kappa\to\infty} \sup_{s\in\mathbb{S}} \frac{1}{\kappa^{\frac{1}{1-\alpha}}\bar{\ell}(\kappa)} \log \mathbb{P}\Big(d\big(\tfrac{1}{\kappa}Z[s, s + \kappa\,\cdot\,], \tfrac{1}{\kappa}Z_{\kappa\cdot}\big) \geqslant 3\delta\Big) \leqslant -\eta.$$

Since $\eta$ and $\delta > 0$ were arbitrary this proves the first statement. □

## 4.2 The large deviation principles

By the exponential equivalence, Proposition 4.4, and (Dembo and Zeitouni, 1998, Theorem 4.2.13) it suffices to prove the large and moderate deviation principles in the framework of the exponentially equivalent processes (15) constructed in the previous section.

The first step in the proof of the first part of Theorem 1.10, is to show a large deviation principle for the occupation times of the underlying process. Throughout this section we denote

$$a_\kappa := \kappa^{1/(1-\alpha)}\bar{\ell}(\kappa).$$

We define the function $\xi\colon \mathbb{R} \to (-\infty, \infty]$ by

$$\xi(u) = \begin{cases} \log \frac{1}{1-u} & \text{if } u < 1, \\ \infty & \text{otherwise.} \end{cases}$$

Its Legendre-Fenchel transform is easily seen to be

$$\xi^*(t) = \begin{cases} t - 1 - \log t & \text{if } t > 0, \\ \infty & \text{otherwise.} \end{cases}$$

**Lemma 4.5.** *For fixed $0 \leqslant u < v$ the family $(\frac{1}{\kappa}T[\kappa u, \kappa v))_{\kappa>0}$ satisfies a large deviation principle with speed $(a_\kappa)$ and rate function $\Lambda^*_{u,v}(t) = \sup_{\zeta\in\mathbb{R}}[t\zeta - \Lambda_{u,v}(\zeta)]$, where*

$$\Lambda_{u,v}(\zeta) = \int_u^v s^{\frac{\alpha}{1-\alpha}}\xi(\zeta s^{-\alpha/(1-\alpha)})\, ds.$$

**Proof.** For fixed $u < v$ denote by $\mathbb{I}_\kappa = \mathbb{I}_\kappa^{[u,v)} = \{j \in \mathbb{Z}_+ : \Phi(j) \in [\kappa u, \kappa v)\}$. We get, using $(S_j)$ for the underlying sequence of $\mathsf{Exp}(f(j))$-distributed independent random variables,

$$\begin{aligned}\Lambda_\kappa(\theta) :=\ & \log \mathbb{E}e^{\theta T[\kappa u, \kappa v)/\kappa} \\ =\ & \sum_{j\in\mathbb{I}_\kappa} \log \mathbb{E}e^{\frac{\theta}{\kappa}S_j} = \sum_{j\in\mathbb{I}_\kappa} \log \frac{1}{1 - \frac{\theta}{\kappa f(j)}} = \sum_{t\in\Phi(\mathbb{I}_\kappa)} \xi\Big(\frac{\theta}{\kappa f(\Phi^{-1}(t))}\Big) \\ =\ & \int_{\bar{\mathbb{I}}_\kappa} f(\Phi^{-1}(t))\, \xi\Big(\frac{\theta}{\kappa f(\Phi^{-1}(t))}\Big)\, dt,\end{aligned}$$

where $\bar{\mathbb{I}}_\kappa = \bar{\mathbb{I}}_\kappa^{[u,v)} = \bigcup_{j\in\mathbb{I}_\kappa}[\Phi(j),\Phi(j+1))$. Now choose $\theta$ in dependence on $\kappa$ as $\theta_\kappa = \zeta\kappa^{1/(1-\alpha)}\bar{\ell}(\kappa)$ with $\zeta < u^{\alpha/(1-\alpha)}$. Then

$$\begin{aligned}\int_{\bar{\mathbb{I}}_\kappa} \bar{f}(t)\xi\Big(\frac{\theta_\kappa}{\kappa\bar{f}(t)}\Big)\,dt &= \kappa\int_{\bar{\mathbb{I}}_\kappa/\kappa} \bar{f}(\kappa s)\xi\Big(\frac{\theta_\kappa}{\kappa\bar{f}(\kappa s)}\Big)\,ds\\ &= \kappa^{1/(1-\alpha)}\int_{\bar{\mathbb{I}}_\kappa/\kappa} s^{\frac{\alpha}{1-\alpha}}\bar{\ell}(\kappa s)\xi\Big(\frac{\zeta\bar{\ell}(\kappa)}{s^{\frac{\alpha}{1-\alpha}}\bar{\ell}(\kappa s)}\Big)\,ds.\end{aligned}$$

Note that $\inf\bar{\mathbb{I}}_\kappa/\kappa$ and $\sup\bar{\mathbb{I}}_\kappa/\kappa$ approach the values $u$ and $v$, respectively. Hence, we conclude with the dominated convergence theorem that one has

$$\Lambda_\kappa(\theta_\kappa) \sim \kappa^{1/(1-\alpha)}\bar{\ell}(\kappa)\underbrace{\int_u^v s^{\frac{\alpha}{1-\alpha}}\xi\big(\tfrac{\zeta}{s^{\frac{\alpha}{1-\alpha}}}\big)\,ds}_{=\Lambda_{u,v}(\zeta)}$$

as $\kappa$ tends to infinity. Now the Gärtner-Ellis theorem implies the large deviation principle for the family $(T[\kappa u,\kappa v))_{\kappa>0}$ for $0<u<v$. It remains to prove the large deviation principle for $u=0$. Note that

$$\mathbb{E}T[0,\kappa v) = \mathbb{E}\sum_{j\in\mathbb{I}_\kappa} S_j = \int_{\bar{\mathbb{I}}_\kappa} f(\Phi^{-1}(t))\frac{1}{f(\Phi^{-1}(t))}\,dt \sim \kappa v$$

and

$$\operatorname{var}(T[0,\kappa v)) = \sum_{j\in\mathbb{I}_\kappa}\operatorname{var}(S_j) = \int_{\bar{\mathbb{I}}_\kappa} f(\Phi^{-1}(t))\frac{1}{f(\Phi^{-1}(t))^2}\,dt \lesssim \frac{1}{f(0)}\kappa v.$$

Consequently, $\frac{T[0,\kappa\varepsilon)}{\kappa}$ converges in probability to $\varepsilon$. Thus for $t<v$

$$\mathbb{P}\Big(\tfrac{1}{\kappa}T[0,\kappa v)\leqslant t\Big) \geqslant \underbrace{\mathbb{P}\Big(\tfrac{1}{\kappa}T[0,\kappa\varepsilon)\leqslant(1+\varepsilon)\varepsilon\Big)}_{\to 1}\,\mathbb{P}\Big(\tfrac{1}{\kappa}T[\kappa\varepsilon,\kappa v)\leqslant t-(1+\varepsilon)\varepsilon\Big)$$

and for sufficiently small $\varepsilon>0$

$$\liminf_{\kappa\to\infty}\frac{1}{a_\kappa}\log\mathbb{P}\Big(\tfrac{1}{\kappa}T[0,\kappa v)\leqslant t\Big) \geqslant -\Lambda^*_{\varepsilon,v}(t-(1+\varepsilon)\varepsilon),$$

while the upper bound is obvious. □

The next lemma is necessary for the analysis of the rate function in Lemma 4.5. It involves the function $\psi$ defined as $\psi(t) = 1-t+t\log t$ for $t\geqslant 0$.

**Lemma 4.6.** *For fixed* $0<x_0<x_1$ *there exists an increasing function* $\eta\colon\mathbb{R}_+\to\mathbb{R}_+$ *with* $\lim_{\delta\downarrow 0}\eta_\delta = 0$ *such that for any* $u,v\in[x_0,x_1]$ *with* $\delta := v-u>0$ *and all* $w\in[u,v], t>0$ *one has*

$$\Big|\Lambda^*_{u,v}(t) - w^{\frac{\alpha}{1-\alpha}}\,t\,\psi\big(\tfrac{\delta}{t}\big)\Big| \leqslant \eta_\delta\Big(\delta + t\,\psi\big(\tfrac{\delta}{t}\big)\Big).$$

We now extend the definition of $\Lambda^*$ continuously by setting, for any $u\geqslant 0$ and $t\geqslant 0$,

$$\Lambda^*_{u,u}(t) = u^{\frac{\alpha}{1-\alpha}}\,t.$$

For the proof of Lemma 4.6 we use the following fact, which can be verified easily.

**Lemma 4.7.** *For any $\zeta > 0$ and $t > 0$, we have $|\xi^*(\zeta t) - \xi^*(t)| \leqslant 2|\zeta - 1| + |\log \zeta| + 2|\zeta - 1|\xi^*(t)$.*

**Proof of Lemma 4.6.** First observe that

$$\gamma_\delta := \sup_{\substack{x_0 < u < v < x_1 \\ v - u \leqslant \delta}} (v/u)^{\frac{\alpha}{1-\alpha}}$$

tends to 1 as $\delta$ tends to zero. By Lemma 4.7, there exists a function $(\bar{\eta}_\delta)_{\delta > 0}$ with $\lim_{\delta \downarrow 0} \bar{\eta}_\delta = 0$ such that for all $\zeta \in [1/\gamma_\delta, \gamma_\delta]$ and $t > 0$

$$|\xi^*(\zeta t) - \xi^*(t)| \leqslant \bar{\eta}_\delta(1 + \xi^*(t)).$$

Consequently, one has for any $\delta > 0$, $x_0 < w, \bar{w} < x_1$ with $|w - \bar{w}| \leqslant \delta$ and $\zeta \in [1/\gamma_\delta, \gamma_\delta]$ that

$$\begin{aligned} |\bar{w}^{\frac{\alpha}{1-\alpha}} \xi^*(\zeta t) - w^{\frac{\alpha}{1-\alpha}} \xi^*(t)| &\leqslant \bar{w}^{\frac{\alpha}{1-\alpha}} |\xi^*(\zeta t) - \xi^*(t)| + \xi^*(t) |\bar{w}^{\frac{\alpha}{1-\alpha}} - w^{\frac{\alpha}{1-\alpha}}| \\ &\leqslant c\bar{\eta}_\delta(1 + \xi^*(t)) + c\delta\xi^*(t), \end{aligned}$$

where $c < \infty$ is a constant only depending on $x_0, x_1$ and $\alpha$. Thus for an appropriate function $(\eta_\delta)_{\delta > 0}$ with $\lim_{\delta \downarrow 0} \eta_\delta = 0$ one gets

$$|\bar{w}^{\frac{\alpha}{1-\alpha}} \xi^*(\zeta t) - w^{\frac{\alpha}{1-\alpha}} \xi^*(t)| \leqslant \eta_\delta(1 + \xi^*(t)). \tag{16}$$

Fix $x_0 < u < v < x_1$ and set $\delta := v - u$. We estimate, for $\theta \geqslant 0$,

$$\delta u^{\frac{\alpha}{1-\alpha}} \xi(\theta v^{-\alpha/(1-\alpha)}) \leqslant \Lambda_{u,v}(\theta) \leqslant \delta v^{\frac{\alpha}{1-\alpha}} \xi(\theta u^{-\alpha/(1-\alpha)}),$$

and the reversed inequalities for $\theta \leqslant 0$. Consequently,

$$\begin{aligned} \Lambda^*_{u,v}(\delta t) &= \sup_\theta [\theta t - \Lambda_{u,v}(\theta)] \\ &\leqslant \delta \sup_\theta [\theta t - u^{\frac{\alpha}{1-\alpha}} \xi(\theta v^{-\alpha/(1-\alpha)})] \vee \delta \sup_\theta [\theta t - v^{\frac{\alpha}{1-\alpha}} \xi(\theta u^{-\alpha/(1-\alpha)})] \\ &= \delta u^{\frac{\alpha}{1-\alpha}} \xi^*((v/u)^{\frac{\alpha}{1-\alpha}} t) \vee \delta v^{\frac{\alpha}{1-\alpha}} \xi^*((u/v)^{\frac{\alpha}{1-\alpha}} t). \end{aligned}$$

Since $(v/u)^{\alpha/(1-\alpha)}$ and $(u/v)^{\alpha/(1-\alpha)}$ lie in $[1/\gamma_\delta, \gamma_\delta]$ we conclude with (16) that for $w \in [u, v)$

$$\Lambda^*_{u,v}(\delta t) \leqslant w^{\frac{\alpha}{1-\alpha}} \xi^*(t)\delta + \eta_\delta(1 + \xi^*(t))\delta.$$

To prove the converse inequality, observe

$$\Lambda^*_{u,v}(t) \geqslant \Big(\delta \sup_{\theta \leqslant 0} [\theta t - u^{\frac{\alpha}{1-\alpha}} \xi(\theta v^{-\alpha/(1-\alpha)})]\Big) \vee \Big(\delta \sup_{\theta \geqslant 0} [\theta t - v^{\frac{\alpha}{1-\alpha}} \xi(\theta u^{-\alpha/(1-\alpha)})]\Big).$$

Now note that the first partial Legendre transform can be replaced by the full one if $t \leqslant (u/v)^{\alpha(1-\alpha)}$. Analogously, the second partial Legendre transform can be replaced if $t \geqslant (v/u)^{\alpha(1-\alpha)}$. Thus we can proceed as above if $t \notin (1/\gamma_\delta, \gamma_\delta)$ and conclude that

$$\Lambda^*_{u,v}(t) \geqslant w^{\frac{\alpha}{1-\alpha}} \xi^*(t)\delta - \eta_\delta(1 + \xi^*(t))\delta.$$

The latter estimate remains valid on $(1/\gamma_\delta, \gamma_\delta)$ if $x_1^{\alpha/(1-\alpha)}(\xi^*(1/\gamma_\delta) \vee \xi^*(\gamma_\delta)) \leqslant \eta_\delta$. Since $\gamma_\delta$ tends to 1 and $\xi^*(1) = 0$ one can make $\eta_\delta$ a bit larger to ensure that the latter estimate is valid and $\lim_{\delta \downarrow 0} \eta_\delta = 0$. This establishes the statement. □

As the next step in the proof of Theorem 1.10 we formulate a finite-dimensional large deviation principle, which can be derived from Lemma 4.5.

**Lemma 4.8.** *Fix $0 = t_0 < t_1 < \cdots < t_p$. Then the vector*

$$\Big(\tfrac{1}{\kappa} Z_{\kappa t_j} \,:\, j \in \{1, \ldots, p\}\Big)$$

*satisfies a large deviation principle in $\{0 \leqslant a_1 \leqslant \cdots \leqslant a_p\} \subset \mathbb{R}^p$ with speed $a_\kappa$ and rate function*

$$J(a_1, \ldots, a_p) = \sum_{j=1}^{p} \Lambda^*_{a_{j-1}, a_j}(t_j - t_{j-1}), \qquad \text{with } a_0 := 0\,.$$

**Proof.** First fix $0 = a_0 < a_1 < \cdots < a_p$. Observe that, whenever $s_{j-1} < s_j$ with $s_0 = 0$,

$$\begin{aligned} &\mathbb{P}\big(\tfrac{1}{\kappa}\, Z_{\kappa t_j} \geqslant a_j > \tfrac{1}{\kappa}\, Z_{\kappa s_j} \text{ for } j \in \{1, \ldots, p\}\big) \\ &\qquad \geqslant \mathbb{P}\big(s_j - s_{j-1} < \tfrac{1}{\kappa} T[a_{j-1}\kappa, a_j\kappa) \leqslant t_j - t_{j-1} \text{ for } j \in \{1, \ldots, p\}\big). \end{aligned}$$

Moreover, supposing that $0 < t_j - t_{j-1} - (s_j - s_{j-1}) \leqslant \delta$ for a $\delta > 0$, we obtain

$$\begin{aligned} &\mathbb{P}\big(a_j \leqslant \tfrac{1}{\kappa} Z_{\kappa t_j} < a_j + \varepsilon \text{ for } j \in \{1, \ldots, p\}\big) \\ &\qquad \geqslant \mathbb{P}\big(\tfrac{1}{\kappa}\, Z_{\kappa t_j} \geqslant a_j > \tfrac{1}{\kappa}\, Z_{\kappa s_j} \text{ and } T[a_j\kappa, (a_j + \varepsilon)\kappa) \geqslant \delta \text{ for } j \in \{1, \ldots, p\}\big) \end{aligned}$$

By Lemma 4.5, given $\varepsilon > 0$ and $A > 0$, we find $\delta > 0$ such that, for $\kappa$ large,

$$\mathbb{P}\big(\tfrac{1}{\kappa} T[a_j\kappa, (a_j + \varepsilon)\kappa) < \delta\big) \leqslant e^{-A a_\kappa}.$$

Hence, for sufficiently small $\delta$ we get with the above estimates that

$$\begin{aligned} &\liminf_{\kappa \to \infty} \frac{1}{a_\kappa} \log \mathbb{P}\big(a_j + \varepsilon > \tfrac{1}{\kappa} Z_{\kappa t_j} \geqslant a_j \text{ for } j \in \{1, \ldots, p\}\big) \\ &\qquad \geqslant \liminf_{\kappa \to \infty} \frac{1}{a_\kappa} \log \mathbb{P}\big(s_j - s_{j-1} < \tfrac{1}{\kappa} T[a_{j-1}\kappa, a_j\kappa) \leqslant t_j - t_{j-1} \text{ for } j \in \{1, \ldots, p\}\big) \\ &\qquad \geqslant -\sum_{j=1}^{p} \Lambda^*_{a_{j-1}, a_j}(t_j - t_{j-1}). \end{aligned}$$

Next, we prove the upper bound. Fix $0 = a_0 \leqslant \ldots \leqslant a_p$ and $0 = b_0 \leqslant \ldots \leqslant b_p$ with $a_j < b_j$, and observe that by the strong Markov property of $(Z_t)$,

$$\begin{aligned} &\mathbb{P}\big(b_j > \tfrac{1}{\kappa}\, Z_{\kappa t_j} \geqslant a_j \text{ for } j \in \{1, \ldots, p\}\big) \\ &\qquad = \prod_{j=1}^{p} \mathbb{P}\big(b_j > \tfrac{1}{\kappa}\, Z_{\kappa t_j} \geqslant a_j \,\big|\, b_i > \tfrac{1}{\kappa}\, Z_{\kappa t_i} \geqslant a_i \text{ for } i \in \{1, \ldots, j-1\}\big) \\ &\qquad \leqslant \prod_{j=1}^{p} \mathbb{P}\big(\tfrac{1}{\kappa} T[b_{j-1}\kappa, a_j\kappa) < t_j - t_{j-1} \leqslant \tfrac{1}{\kappa} T[a_{j-1}\kappa, b_j\kappa)\big). \end{aligned}$$

Consequently,

$$\limsup_{\kappa \uparrow \infty} \frac{1}{a_\kappa} \log \mathbb{P}\big(b_j > \tfrac{1}{\kappa}\, Z_{\kappa t_j} \geqslant a_j \text{ for } j \in \{1, \ldots, p\}\big) \leqslant -\sum_{j=1}^{p} r_j,$$

where

$$r_j = \begin{cases} \Lambda^*_{b_{j-1},a_j}(t_j - t_{j-1}) & \text{if } a_j - b_{j-1} \geqslant t_j - t_{j-1}, \\ \Lambda^*_{a_{j-1},b_j}(t_j - t_{j-1}) & \text{if } b_j - a_{j-1} \leqslant t_j - t_{j-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Using the continuity of $(u,v) \mapsto \Lambda^*_{u,v}(t)$ for fixed $t$, it is easy to verify continuity of each $r_j$ of the parameters $a_{j-1}$, $a_j$, $b_{j-1}$, and $b_j$. Suppose now that $(a_j)$ and $(b_j)$ are taken from a predefined compact subset of $\mathbb{R}^d$. Then we have

$$\sum_{j=1}^{p} \left| r_j - \Lambda^*_{a_{j-1},a_j}(t_j - t_{j-1}) \right| \leqslant \vartheta\big( \max\{b_j - a_j : j = 1, \ldots, p\}\big),$$

for an appropriate function $\vartheta$ with $\lim_{\delta \downarrow 0} \vartheta(\delta) = 0$. Now the upper bound follows with an obvious exponential tightness argument. □

We can now prove a large deviation principle in a weaker topology, by taking a projective limit and simplifying the resulting rate function with the help of Lemma 4.6.

**Lemma 4.9.** *On the space of increasing functions with the topology of pointwise convergence the family of functions*

$$\Big( \tfrac{1}{\kappa} Z_{\kappa t} \colon t \geqslant 0 \Big)_{\kappa > 0}$$

*satisfies a large deviation principle with speed $(a_\kappa)$ and rate function $J$.*

**Proof.** Observe that the space of increasing functions equipped with the topology of pointwise convergence can be interpreted as projective limit of the spaces $\{0 \leqslant a_1 \leqslant \cdots \leqslant a_p\}$ with the canonical projections given by $\pi(x) = (x(t_1), \ldots, x(t_p))$ for $0 < t_1 < \ldots < t_p$. By the Dawson-Gärtner theorem, we obtain a large deviation principle with good rate function

$$\tilde{J}(x) = \sup_{0 < t_1 < \ldots < t_p} \sum_{j=1}^{p} \Lambda^*_{x(t_{j-1}),x(t_j)}(t_j - t_{j-1}).$$

Note that the value of the variational expression is nondecreasing, if additional points are added to the partition. It is not hard to see that $\tilde{J}(x) = \infty$, if $x$ fails to be absolutely continuous. Indeed, there exists $\delta > 0$ and, for every $n \in \mathbb{N}$, a partition $\delta \leqslant s_1^n < t_1^n \leqslant \cdots \leqslant s_n^n < t_n^n \leqslant \frac{1}{\delta}$ such that $\sum_{j=1}^n t_j^n - s_j^n \to 0$ but $\sum_{j=1}^n x(t_j^n) - x(s_j^n) \geqslant \delta$. Then, for any $\lambda > 0$,

$$\begin{aligned} \tilde{J}(x) &= \sup_{\substack{0 < t_1 < \ldots < t_p \\ \lambda_1, \ldots, \lambda_p \in \mathbb{R}}} \sum_{j=1}^{p} \lambda_j \big(t_j - t_{j-1}\big) - \Lambda_{x(t_{j-1}),x(t_j)}(\lambda_j) \\ &\geqslant \sum_{j=1}^{n} \Big[ -\lambda\big(t_j^n - s_j^n\big) + \int_{x(s_j^n)}^{x(t_j^n)} u^{\frac{\alpha}{1-\alpha}} \log\big(1 + \lambda u^{\frac{-\alpha}{1-\alpha}}\big)\, du \Big] \\ &\geqslant -\lambda \sum_{j=1}^{n} \big(t_j^n - s_j^n\big) + \delta^{\frac{1}{1-\alpha}} \log\big(1 + \lambda \delta^{\frac{\alpha}{1-\alpha}}\big) \longrightarrow \delta^{\frac{1}{1-\alpha}} \log\big(1 + \lambda \delta^{\frac{\alpha}{1-\alpha}}\big), \end{aligned}$$

which can be made arbitrarily large by choice of $\lambda$.

From now on suppose that $x$ is absolutely continuous. The remaining proof is based on the equation

$$\tilde{J}(x) = \sup_{0<t_1<\ldots<t_p} \sum_{j=1}^{p} \big(t_j - t_{j-1}\big)\, x(t_j)^{\frac{\alpha}{1-\alpha}}\, \psi\Big(\frac{x(t_j) - x(t_{j-1})}{t_j - t_{j-1}}\Big). \tag{17}$$

Before we prove its validity we apply (17) to derive the assertions of the lemma. For the *lower bound* we choose a scheme $0 < t_1^n < \cdots < t_n^p$, with $p$ depending on $n$, such that $t_n^p \to \infty$ and the mesh goes to zero. Define, for $t_{j-1}^n \leqslant t < t_j^n$,

$$x_j^n(t) = \frac{1}{t_j^n - t_{j-1}^n} \int_{t_{j-1}^n}^{t_j^n} \dot{x}_s\, ds = \frac{x(t_j^n) - x(t_{j-1}^n)}{t_j^n - t_{j-1}^n}.$$

Note that, by Lebesgue's theorem, $x_j^n(t) \to \dot{x}_t$ almost everywhere. Hence

$$\tilde{J}(x) \geqslant \liminf_{n\to\infty} \int_0^{t_p^n} x_t^{\frac{\alpha}{1-\alpha}}\, \psi\big(x_j^n(t)\big)\, dt \geqslant \int_0^\infty x_t^{\frac{\alpha}{1-\alpha}} \liminf_{n\to\infty} \psi\big(x_j^n(t)\big)\, dt = J(x).$$

For the *upper bound* we use the convexity of $\psi$ to obtain

$$\psi\Big(\frac{x(t_j) - x(t_{j-1})}{t_j - t_{j-1}}\Big) = \psi\Big(\frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \dot{x}_t\, dt\Big) \leqslant \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \psi\big(\dot{x}_t\big)\, dt.$$

Hence

$$\tilde{J}(x) \leqslant \sup_{0<t_1<\ldots<t_p} \sum_{j=1}^{p} x(t_j)^{\frac{\alpha}{1-\alpha}} \int_{t_{j-1}}^{t_j} \psi\big(\dot{x}_t\big)\, dt = J(x)\,,$$

as required to complete the proof.
It remains to prove (17). We fix $t'$ and $t''$ with $t' < t''$ and $x(t') > 0$, and partitions $t' = t_0^n < \cdots < t_n^n = t''$ with $\delta_n := \sup_j x(t_j^n) - x(t_{j-1}^n)$ converging to 0. Assume $n$ is sufficiently large such that $\eta_{\delta_n} \leqslant \frac{1}{2}(t')^{\frac{\alpha}{1-\alpha}}$, with $\eta$ as in Lemma 4.6. Then,

$$\begin{aligned} &\sum_{j=1}^{n} \Lambda^*_{x(t_{j-1}^n),x(t_j^n)}(t_j^n - t_{j-1}^n) \\ &\qquad \geqslant \frac{1}{2}(t')^{\frac{\alpha}{1-\alpha}} \Big[\underbrace{\sum_{j=1}^{n} (t_j^n - t_{j-1}^n)\, \psi\big(\frac{x(t_j^n) - x(t_{j-1}^n)}{t_j^n - t_{j-1}^n}\big)}_{(*)} - (x(t'') - x(t'))\Big], \end{aligned} \tag{18}$$

and $(*)$ is uniformly bounded as long as $\tilde{J}(x)$ is finite. On the other hand also the finiteness of the right hand side of (17) implies uniform boundedness of $(*)$. Hence, either both expressions in (17) are infinite or we conclude with Lemma 4.6 that for an appropriate choice of $t_j^n$,

$$\begin{aligned} \sup_{t'=t_0<\cdots<t_p=t''} \sum_{j=1}^{p} \Lambda^*_{x(t_{j-1}),x(t_j)}\big(t_j - t_{j-1}\big) &= \lim_{n\to\infty} \sum_{j=1}^{n} \Lambda^*_{x(t_{j-1}^n),x(t_j^n)}\big(t_j^n - t_{j-1}^n\big) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \big(t_j^n - t_{j-1}^n\big)\, x(t_j^n)^{\frac{\alpha}{1-\alpha}}\, \psi\Big(\frac{x(t_j^n) - x(t_{j-1}^n)}{t_j^n - t_{j-1}^n}\Big) \\ &= \sup_{t'=t_0<\cdots<t_p=t''} \sum_{j=1}^{p} \big(t_j - t_{j-1}\big)\, x(t_j)^{\frac{\alpha}{1-\alpha}}\, \psi\Big(\frac{x(t_j) - x(t_{j-1})}{t_j - t_{j-1}}\Big). \end{aligned}$$

This expression easily extends to formula (17). $\square$

**Lemma 4.10.** *The level sets of $J$ are compact in $\mathcal{I}[0,\infty)$.*

**Proof.** We have to verify the assumptions of the Arzelà-Ascoli theorem. Fix $\delta \in (0,1)$, $t \geqslant 0$, and a function $x \in \mathcal{I}[0,\infty)$ with finite rate $J$. We choose $\delta' \in (0,\delta)$ with $x_{t+\delta'} = \frac{1}{2}(x_t + x_{t+\delta})$, denote $\varepsilon = x_{t+\delta} - x_t$, and observe that

$$\begin{aligned} J(x) &\geqslant \int_t^{t+\delta} x_s^{\frac{\alpha}{1-\alpha}} [1 - \dot{x}_s + \dot{x}_s \log \dot{x}_s] \, ds \\ &\geqslant (\delta - \delta') \Big(\frac{\varepsilon}{2}\Big)^{\frac{\alpha}{1-\alpha}} \int_{t+\delta'}^{t+\delta} [1 - \dot{x}_s + \dot{x}_s \log \dot{x}_s] \, \frac{ds}{\delta - \delta'}. \end{aligned}$$

Here we used that $x_s \geqslant \varepsilon/2$ for $s \in [t+\delta', t+\delta]$. Next, we apply Jensen's inequality to the convex function $\psi$ to deduce that

$$J(x) \geqslant (\delta - \delta') \Big(\frac{\varepsilon}{2}\Big)^{\frac{\alpha}{1-\alpha}} \psi\Big(\frac{1}{\delta - \delta'} \frac{\varepsilon}{2}\Big).$$

Now assume that $\frac{\varepsilon}{2} \geqslant \delta$. Elementary calculus yields

$$J(x) \geqslant \delta \Big(\frac{\varepsilon}{2}\Big)^{\frac{\alpha}{1-\alpha}} \psi\Big(\frac{1}{\delta} \frac{\varepsilon}{2}\Big) \geqslant \Big(\frac{\varepsilon}{2}\Big)^{\frac{1}{1-\alpha}} \log \frac{\varepsilon}{2e\delta}.$$

If we additionally assume $\varepsilon \geqslant 2e\delta^{\frac{1}{2}}$, then we get $(J(x)/\log \delta^{-\frac{1}{2}})^{1-\alpha} \geqslant \varepsilon$. Therefore, in general

$$x_{t+\delta} - x_t \leqslant \max\Big(2\Big(\frac{J(x)}{\log \delta^{-\frac{1}{2}}}\Big)^{1-\alpha}, 2e\delta^{\frac{1}{2}}\Big).$$

Hence the level sets are uniformly equicontinuous. As $x_0 = 0$ for all $x \in \mathcal{I}[0,\infty)$ this implies that the level sets are uniformly bounded on compact sets, which finishes the proof. $\square$

We now improve our large deviation principle to the topology of uniform convergence on compact sets, which is stronger than the topology of pointwise convergence. To this end we introduce, for every $m \in \mathbb{N}$, a mapping $f_m$ acting on functions $x \colon [0,\infty) \to \mathbb{R}$ by

$$f_m(x)_t = x_{t_j} \qquad \text{if } t_j := \tfrac{j}{m} \leqslant t < \tfrac{j+1}{m} =: t_{j+1}. \tag{19}$$

**Lemma 4.11.** *For every $\delta > 0$ and $T > 0$, we have*

$$\lim_{m\to\infty} \limsup_{\kappa\uparrow\infty} \frac{1}{a_\kappa} \log \mathbb{P}\Big( \sup_{0\leqslant t\leqslant T} \big| f_m\big(\tfrac{1}{\kappa} Z_{\kappa\,\cdot}\big)_t - \tfrac{1}{\kappa} Z_{\kappa t}\big| > \delta\Big) = -\infty.$$

**Proof.** Note that

$$\mathbb{P}\Big( \sup_{0\leqslant t\leqslant T} \big| f_m\big(\tfrac{1}{\kappa} Z_{\kappa\,\cdot}\big)_t - \tfrac{1}{\kappa} Z_{\kappa t}\big| \geqslant \delta\Big) \leqslant \sum_{j=0}^{Tm} \mathbb{P}\big(\tfrac{1}{\kappa} Z_{\kappa t_{j+1}} - \tfrac{1}{\kappa} Z_{\kappa t_j} \geqslant \delta\big).$$

By Lemma 4.9 we have

$$\limsup_{\kappa\uparrow\infty} \frac{1}{a_\kappa} \log \mathbb{P}\big(\tfrac{1}{\kappa} Z_{\kappa t_{j+1}} - \tfrac{1}{\kappa} Z_{\kappa t_j} \geqslant \delta\big) \leqslant \inf\big\{J(x) \colon x_{t_{j+1}} - x_{t_j} \geqslant \delta\big\},$$

and, by Lemma 4.10, the right hand side diverges to infinity, uniformly in $j$, as $m \uparrow \infty$. $\square$

**Proof of the first large deviation principle in Theorem 1.10.** We apply (Dembo and Zeitouni, 1998, Theorem 4.2.23), which allows to transfer the large deviation principle from the topological Hausdorff space of increasing functions with the topology of pointwise convergence, to the metrizable space $\mathcal{I}[0,\infty)$ by means of the sequence $f_m$ of continuous mappings approximating the identity. Two conditions need to be checked: On the one hand, using the equicontinuity of the sets $\{I(x) \leqslant \eta\}$ established in Lemma 4.10, we easily obtain

$$\limsup_{m\to\infty} \sup_{J(x)\,\leqslant\,\eta} d\big(f_m(x), x\big) = 0,$$

for every $\eta > 0$, where $d$ denotes a suitable metric on $\mathcal{I}[0,\infty)$. On the other hand, by Lemma 4.11, we have that $(f_m(\frac{1}{\kappa}Z_{\kappa}\cdot))$ are a family of exponentially good approximations of $(\frac{1}{\kappa}Z_{\kappa}\cdot)$. $\square$

The proof of the second large principle can be done from first principles.

**Proof of the second large deviation principle in Theorem 1.10.** For the *lower* bound observe that, for any $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} |\tfrac{1}{\kappa}Z_{\kappa t} - (t-a)_+| < \varepsilon\Big) \geqslant \mathbb{P}\big(Z_{\kappa a} = 0\big)\, \mathbb{P}\Big(\sup_{a\leqslant t\leqslant T} |\tfrac{1}{\kappa}(Z_{\kappa t} - Z_{\kappa a}) - (t-a)| < \varepsilon\Big),$$

and recall that the first probability on the right hand side is $\exp\{-\kappa\, a f(0)\}$ and the second converges to one, by the law of large numbers. For the *upper* bound note first that, by the first large deviation principle, for any $\varepsilon > 0$ and closed set $A \subset \{J(x) > \varepsilon\}$,

$$\limsup_{\kappa\uparrow\infty} \frac{1}{\kappa} \log \mathbb{P}\big(\tfrac{1}{\kappa}Z_{\kappa}\cdot \in A\big) = -\infty.$$

Note further that, for any $\delta > 0$ and $T > 0$, there exists $\varepsilon > 0$ such that $J(x) \leqslant \varepsilon$ implies $\sup_{0\leqslant t\leqslant T} |x - y| < \delta$, where $y_t = (t-a)_+$ for some $a \in [0,T]$. Then, for $\theta < f(0)$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} |\tfrac{1}{\kappa}Z_{\kappa t} - y| \leqslant \delta\Big) &\leqslant \mathbb{P}\big(Z_{\kappa a} \leqslant \delta\kappa\big) = \mathbb{P}\big(T[0,\kappa\delta] \geqslant \kappa a\big) \leqslant e^{-\kappa a\theta} \prod_{\Phi(j)\leqslant\kappa\delta} \mathbb{E}\exp\{\theta S_j\} \\ &= e^{-\kappa a\theta} \exp \sum_{\Phi(j)\leqslant\kappa\delta} \log \frac{1}{1 - \frac{\theta}{f(j)}},\end{aligned}$$

and the result follows because the sum on the right is bounded by a constant multiple of $\kappa\delta$. $\square$

### 4.3 The moderate deviation principle

Recall from the beginning of Section 4.2 that it is sufficient to show Theorem 1.12 for the approximating process $Z$ defined in (15). We initially include the case $c = \infty$ in our consideration, and abbreviate

$$b_\kappa := a_\kappa\, \kappa^{\frac{2\alpha-1}{1-\alpha}}\, \bar\ell(\kappa) \ll \kappa^{\frac{\alpha}{1-\alpha}}\, \bar\ell(\kappa),$$

so that we are looking for a moderate deviation principle with speed $a_\kappa b_\kappa$.

**Lemma 4.12.** *Let $0 \leqslant u < v$, suppose that $f$ and $a_\kappa$ are as in Theorem 1.12 and define*

$$\mathcal{I}_{[u,v)} = \int_u^v s^{-\frac{\alpha}{1-\alpha}}\, ds = \tfrac{1-\alpha}{1-2\alpha}\left(v^{\frac{1-2\alpha}{1-\alpha}} - u^{\frac{1-2\alpha}{1-\alpha}}\right).$$

*Then the family*

$$\left(\frac{T[\kappa u, \kappa v) - \kappa(v-u))}{a_\kappa}\right)_{\kappa>0}$$

*satisfies a large deviation principle with speed $(a_\kappa b_\kappa)$ and rate function*

$$I_{[u,v)}(t) = \begin{cases} \frac{1}{2\mathcal{I}_{[u,v)}}\, t^2 & \text{if } u > 0 \text{ or } t \leqslant \frac{1}{c}\,\mathcal{I}_{[0,v)}\, f(0), \\ \frac{1}{c}\, f(0)\, t - \frac{1}{2}\mathcal{I}_{[0,v)}(\frac{1}{c}\, f(0))^2 & \text{if } u = 0 \text{ and } t \geqslant \frac{1}{c}\,\mathcal{I}_{[0,v)}\, f(0). \end{cases}$$

**Proof.** Denoting by $\Lambda_\kappa$ the logarithmic moment generating function of $b_\kappa\,(T[\kappa u, \kappa v) - \kappa(v-u))$, observe that

$$\Lambda_\kappa(\theta) = \log \mathbb{E} \exp\big\{\theta b_\kappa\,\big(T[\kappa u, \kappa v) - \kappa(v-u)\big)\big\} = \sum_{w \in \mathbb{S} \cap [\kappa u, \kappa v)} \log \mathbb{E} \exp\big\{\theta b_\kappa\,\big(T[w]\big)\big\} - \theta \kappa b_\kappa (v-u)$$

$$= \sum_{w \in \mathbb{S} \cap [\kappa u, \kappa v)} \xi\Big(\frac{\theta b_\kappa}{\bar f(w)}\Big) - \theta \kappa b_\kappa\,(v-u) = \int_{\mathbb{I}_\kappa} \bar f(w) \xi\Big(\frac{\theta b_\kappa}{\bar f(w)}\Big)\, dw - \theta \kappa b_\kappa\,(v-u), \tag{20}$$

where $\mathbb{I}_\kappa = \{w \geqslant 0 : \iota(w) \in [\kappa u, \kappa v)\}$ and $\iota(w) = \max \mathbb{S} \cap [0, w]$. Since $\kappa u \leqslant \inf \mathbb{I}_\kappa < \kappa u + (f(0))^{-1}$ and $\kappa v \leqslant \sup \mathbb{I}_\kappa < \kappa v + (f(0))^{-1}$ we get

$$\Big|\Lambda_\kappa(\theta) - \int_{\mathbb{I}_\kappa} \big[\bar f(w) \xi\Big(\frac{\theta b_\kappa}{\bar f(w)}\Big) - \theta b_\kappa\big]\, dw\Big| \leqslant \frac{2\theta b_\kappa}{f(0)}. \tag{21}$$

Now focus on the case $u > 0$. A Taylor approximation gives $\xi(w) = w + \frac{1}{2}(1 + o(1)) w^2$, as $w \downarrow 0$. By dominated convergence,

$$\begin{aligned} \int_{\mathbb{I}_\kappa} \big[\bar f(w) \xi\Big(\frac{\theta b_\kappa}{\bar f(w)}\Big) - \theta b_\kappa\big]\, dw &\sim \frac{1}{2} \int_{\mathbb{I}_\kappa} \frac{1}{\bar f(w)}\, dw \times \theta^2 b_\kappa^2 \\ &\sim \frac{1}{2}\, \frac{\kappa^{\frac{1-2\alpha}{1-\alpha}}}{\bar\ell(\kappa)} \int_u^v w^{-\frac{\alpha}{1-\alpha}}\, dw \times \theta^2 b_\kappa^2 \\ &= a_\kappa b_\kappa\, \frac{1}{2}\, \mathcal{I}_{[u,v)}\, \theta^2\,. \end{aligned}$$

Together with (21) we arrive at

$$\Lambda_\kappa(\theta) \sim a_\kappa b_\kappa\, \frac{1}{2}\, \mathcal{I}_{[u,v)} \theta^2.$$

Now the Gärtner-Ellis theorem implies that the family $((T[\kappa u, \kappa v) - \kappa(v-u))/a_\kappa)$ satisfies a large deviation principle with speed $(a_\kappa b_\kappa)$ having as rate function the Fenchel-Legendre transform of $\frac{1}{2}\mathcal{I}_{[u,v)}\theta^2$ which is $I_{[u,v)}$.

Next, we look at the case $u = 0$. If $\theta \geqslant \frac{1}{c}\, f(0)$ then $\Lambda_\kappa(\theta) = \infty$ for all $\kappa > 0$, so assume the contrary. The same Taylor expansion as above now gives

$$\bar f(w) \xi\Big(\frac{\theta b_\kappa}{\bar f(w)}\Big) - \theta b_\kappa \sim \frac{1}{2}\, \frac{\theta^2 b_\kappa^2}{\bar f(w)}$$

as $w \uparrow \infty$. In particular, the integrand in (20) is regularly varying with index $-\frac{\alpha}{1-\alpha} > -1$ and we get from Karamata's theorem, see e.g. (Bingham et al., 1987, Theorem 1.5.11), that

$$\Lambda_\kappa(\theta) \sim \frac{1}{2}\,\theta^2 b_\kappa^2\,\frac{\kappa^{\frac{1-2\alpha}{1-\alpha}}}{\bar\ell(\kappa)}\int_0^v s^{-\alpha/(1-\alpha)}\,ds = a_\kappa b_\kappa\,\frac{1}{2}\,\mathcal{I}_{[0,v)}\,\theta^2. \tag{22}$$

Consequently,

$$\lim_{\kappa\to\infty}\frac{1}{a_\kappa b_\kappa}\Lambda_\kappa(\theta) = \begin{cases} \frac{1}{2}\,\mathcal{I}_{[0,v)}\theta^2 & \text{if } \theta < \frac{1}{c}\,f(0),\\ \infty & \text{otherwise.}\end{cases}$$

The Legendre transform of the right hand side is

$$I_{[0,v)}(t) = \begin{cases} \frac{1}{2\mathcal{I}_{[0,v)}}t^2 & \text{if } t \leqslant \frac{1}{c}\,\mathcal{I}_{[0,v)}\,f(0),\\ \frac{1}{c}\,f(0)\,t - \frac{1}{2}\mathcal{I}_{[0,v)}\big(\frac{1}{c}\,f(0)\big)^2 & \text{if } t \geqslant \frac{1}{c}\,\mathcal{I}_{[0,v)}\,f(0).\end{cases}$$

Since $I_{[0,v)}$ is not strictly convex the Gärtner-Ellis Theorem does not imply the full large deviation principle. It remains to prove the lower bound for open sets $(t,\infty)$ with $t \geqslant \frac{1}{c}\,\mathcal{I}_{[0,v)}f(0)$. Fix $\varepsilon \in (0,u)$ and note that, for sufficiently large $\kappa$,

$$\begin{aligned}\mathbb{P}\big((T[\kappa u,\kappa v)-\kappa v)/a_\kappa > t\big) &\geqslant \mathbb{P}\big((T[\kappa\varepsilon,\kappa v)-\kappa(v-\varepsilon))/a_\kappa > \tfrac{1}{c}\,\mathcal{I}_{[0,v)}f(0)\big)\\ &\quad\times \underbrace{\mathbb{P}\big((T(0,\kappa\varepsilon)-\kappa\varepsilon)/a_\kappa > -\varepsilon\big)}_{\to 1}\,\mathbb{P}\big(T[0]/a_\kappa > t - \tfrac{1}{c}\,\mathcal{I}_{[0,v)}f(0)+\varepsilon\big).\end{aligned}$$

so that by the large deviation principle for $((T_{[\kappa\varepsilon,\kappa v)} - \kappa(v-\varepsilon))/a_\kappa)$ and the exponential distribution it follows that

$$\liminf_{\kappa\to\infty}\frac{1}{a_\kappa b_\kappa}\log\mathbb{P}\big((T[0,\kappa v)-\kappa v)/a_\kappa > t\big) \geqslant -\frac{(\frac{1}{c}\,\mathcal{I}_{[0,v)}f(0))^2}{2\mathcal{I}_{[\varepsilon,v)}} - (t - \tfrac{1}{c}\,\mathcal{I}_{[0,v)}f(0)+\varepsilon)\tfrac{1}{c}\,f(0).$$

Note that the right hand side converges to $-I_{[0,v)}(t)$ when letting $\varepsilon$ tend to zero. This establishes the full large deviation principle for $((T[0,\kappa v)-\kappa v)/a_\kappa)$. □

We continue the proof of Theorem 1.12 with a finite-dimensional moderate deviation principle, which can be derived from Lemma 4.12.

**Lemma 4.13.** *Fix $0 = t_0 < t_1 < \cdots < t_p$. Then the vector*

$$\Big(\tfrac{1}{a_\kappa}\big(Z_{\kappa t_j} - \kappa t_j\big)\ :\ j \in \{1,\dots,p\}\Big)$$

*satisfies a large deviation principle in $\mathbb{R}^p$ with speed $a_\kappa b_\kappa$ and rate function*

$$I(a_1,\dots,a_p) = \sum_{j=1}^p I_{[t_{j-1},t_j)}(a_{j-1}-a_j), \qquad \textit{with } a_0 := 0\,.$$

**Proof.** We note that, for $-\infty \leqslant a^{(j)} < b^{(j)} \leqslant \infty$, we have (interpreting conditions on the right as void, if they involve infinity)

$$\begin{aligned}&\mathbb{P}\big(a^{(j)}\,a_\kappa \leqslant Z_{\kappa t_j} - \kappa t_j < b^{(j)}\,a_\kappa \text{ for all } j\big)\\ &\quad= \mathbb{P}\big(T[0,\kappa t_j + a_\kappa a^{(j)}) \leqslant \kappa\,t_j, T[0,\kappa t_j + a_\kappa b^{(j)}) > \kappa\,t_j \text{ for all } j\big)\,.\end{aligned}$$

To continue from here we need to show that the random variables $T[0,\kappa t+a_\kappa b)$ and $T[0,\kappa t)+a_\kappa b$ are exponentially equivalent in the sense that

$$\lim_{\kappa\to\infty} a_\kappa^{-1} b_\kappa^{-1} \log \mathbb{P}\big(\big|T[0,\kappa t+a_\kappa b)-T[0,\kappa t)-a_\kappa b\big|>a_\kappa\varepsilon\big)=-\infty\,. \tag{23}$$

Indeed, first let $b>0$. As in Lemma 4.12, we see that for any $t\geqslant 0$ and $\theta\in\mathbb{R}$,

$$a_\kappa^{-1} b_\kappa^{-1} \,\log \mathbb{E}\exp\big\{\theta b_\kappa\big(T[\kappa t,\kappa t+a_\kappa b)-a_\kappa b\big)\big\}\longrightarrow 0, \tag{24}$$

Chebyshev's inequality gives, for any $A>0$,

$$\begin{aligned}\mathbb{P}\big(T[0,&\kappa t+a_\kappa b)-T[0,\kappa t)-a_\kappa b>a_\kappa\varepsilon\big)\\ &\leqslant e^{-Aa_\kappa b_\kappa}\,\mathbb{E}\exp\big\{\tfrac{A}{\varepsilon}\,b_\kappa\big(T[\kappa t,\kappa t+a_\kappa b)-a_\kappa b\big)\big\}.\end{aligned}$$

A similar estimate can be performed for $\mathbb{P}(T[0,\kappa t+a_\kappa b)-T[0,\kappa t)-a_\kappa b<-a_\kappa\varepsilon)$, and the argument also extends to the case $b<0$. From this (23) readily follows.

Using Lemma 1.12 and independence, we obtain a large deviation principle for the vector

$$\Big(\tfrac{1}{a_\kappa}\big(T[\kappa t_{j-1},\kappa t_j)-\kappa\,(t_j-t_{j-1})\big)\;:\;j\in\{1,\dots,p\}\Big),$$

with rate function

$$I_1(a_1,\dots,a_p)=\sum_{j=1}^p I_{[t_{j-1},t_j)}(a_j).$$

Using the contraction principle, we infer from this a large deviation principle for the vector

$$\Big(\tfrac{1}{a_\kappa}\big(T[0,\kappa t_j)-\kappa\,t_j\big)\;:\;j\in\{1,\dots,p\}\Big)$$

with rate function

$$I_2(a_1,\dots,a_p)=\sum_{j=1}^p I_{[t_{j-1},t_j)}(a_j-a_{j-1})\,.$$

Combining this with (23) we obtain that

$$\begin{aligned}a_\kappa^{-1}b_\kappa^{-1}\log\mathbb{P}\big(T[0,\kappa t_j+a_\kappa a^{(j)}]<\kappa\,t_j, T[0,\kappa t_j+a_\kappa b^{(j)}]>\kappa\,t_j \text{ for all } j\big)\\ \sim a_\kappa^{-1}b_\kappa^{-1}\log\mathbb{P}\big(-a_\kappa b^{(j)}<T[0,\kappa t_j]-\kappa\,t_j<-a_\kappa a^{(j)} \text{ for all } j\big),\end{aligned}$$

and (observing the signs!) the required large deviation principle. □

We may now take a projective limit and arrive at a large deviation principle in the space $\mathcal{P}(0,\infty)$ of functions $x\colon(0,\infty)\to\mathbb{R}$ equipped with the topology of pointwise convergence.

**Lemma 4.14.** *The family of functions*

$$\Big(\tfrac{1}{a_\kappa}\big(Z_{\kappa t}-\kappa t\big)\colon t>0\Big)_{\kappa>0}$$

*satisfies a large deviation principle in the space $\mathcal{P}(0,\infty)$, with speed $a_\kappa b_\kappa$ and rate function*

$$I(x)=\begin{cases}\frac12\int_0^\infty(\dot x_t)^2\,t^{\frac{\alpha}{1-\alpha}}\,dt-\frac1c\,f(0)\,x_0 & \textit{if } x \textit{ is absolutely continuous and } x_0\leqslant 0.\\ \infty & \textit{otherwise.}\end{cases}$$

**Proof.** Observe that the space of functions equipped with the topology of pointwise convergence can be interpreted as the projective limit of $\mathbb{R}^p$ with the canonical projections given by $\pi(x) = (x(t_1), \dots, x(t_p))$ for $0 < t_1 < \dots < t_p$. By the Dawson-Gärtner theorem, we obtain a large deviation principle with rate function

$$\tilde{I}(x) = \sup_{0<t_1<\dots<t_p} \sum_{j=2}^{p} I_{[t_{j-1},t_j)}\big(x_{t_{j-1}} - x_{t_j}\big) + I_{[0,t_1)}\big(-x_{t_1}\big).$$

Note that the value of the variational expression is nondecreasing, if additional points are added to the partition. We first fix $t_1 > 0$ and optimize the first summand independently. Observe that

$$\sup_{t_1<\dots<t_p} \sum_{j=2}^{p} I_{[t_{j-1},t_j)}\big(x_{t_{j-1}} - x_{t_j}\big) = \tfrac{1}{2} \sup_{t_1<t_2<\dots<t_p} \sum_{j=2}^{p} \frac{\big(x_{t_j} - x_{t_{j-1}}\big)^2}{\frac{1-\alpha}{1-2\alpha}\big(t_j^{\frac{1-2\alpha}{1-\alpha}} - t_{j-1}^{\frac{1-2\alpha}{1-\alpha}}\big)}.$$

Recall that

$$\big(t_j - t_{j-1}\big)\, t_j^{\frac{-\alpha}{1-\alpha}} \leqslant \tfrac{1-\alpha}{1-2\alpha}\big(t_j^{\frac{1-2\alpha}{1-\alpha}} - t_{j-1}^{\frac{1-2\alpha}{1-\alpha}}\big) \leqslant \big(t_j - t_{j-1}\big)\, t_{j-1}^{\frac{-\alpha}{1-\alpha}}.$$

Hence we obtain an upper and [in brackets] lower bound of

$$\tfrac{1}{2} \sup_{t_1<t_2<\dots<t_p} \sum_{j=2}^{p} \Big(\frac{x_{t_j} - x_{t_{j-1}}}{t_j - t_{j-1}}\Big)^2 t_{j[-1]}^{\frac{\alpha}{1-\alpha}} \big(t_j - t_{j-1}\big).$$

It is easy to see that (using arguments analogous to those given in the last step in the proof of the first large deviation principle) that this is $+\infty$ if $x$ fails to be absolutely continuous, and otherwise it equals

$$\tfrac{1}{2} \int_{t_1}^{\infty} (\dot{x}_t)^2\, t^{\frac{\alpha}{1-\alpha}}\, dt.$$

In the latter case we have

$$\tilde{I}(x) = \lim_{t_1 \downarrow 0} \tfrac{1}{2} \int_{t_1}^{\infty} (\dot{x}_t)^2\, t^{\frac{\alpha}{1-\alpha}}\, dt + I_{[0,t_1)}\big(-x_{t_1}\big).$$

If $x_0 > 0$ the last summand diverges to infinity. If $x_0 = 0$ and the limit of the integral is finite, then using Cauchy-Schwarz,

$$I_{[0,t_1)}\big(-x_{t_1}\big) \leqslant \tfrac{1}{2}\mathcal{I}_{[0,t_1)}^{-1} \Big| \int_0^{t_1} \dot{x}_t\, dt \Big|^2 \leqslant \tfrac{1}{2} \int_0^{\varepsilon} (\dot{x}_t)^2 t^{\frac{\alpha}{1-\alpha}}\, dt,$$

hence it converges to zero. If $x_0 < 0$,

$$\lim_{t_1 \downarrow 0} I_{[0,t_1)}\big(-x_{t_1}\big) = \lim_{t_1 \downarrow 0} -\tfrac{1}{c}\, f(0)\, x_{t_1} + \tfrac{1-\alpha}{1-2\alpha}\, t_1^{\frac{\alpha}{1-\alpha}} \big(\tfrac{1}{c}\, f(0)\big)^2 = -\tfrac{1}{c}\, f(0)\, x_0,$$

as required to complete the proof. □

**Lemma 4.15.** *If $c < \infty$, the function $I$ is a good rate function on $\mathcal{L}(0, \infty)$.*

**Proof.** Recall that, by the Arzelà-Ascoli theorem, it suffices to show that for any $\eta > 0$ the family $\{x \colon I(x) \leqslant \eta\}$ is bounded and equicontinuous on every compact subset of $(0, \infty)$.

Suppose that $I(x) \leqslant \eta$ and $0 < s < t$. Then, using Cauchy-Schwarz in the second step,

$$
\begin{aligned}
|x_t - x_s| &= \Big| \int_s^t \dot{x}_u \, du \Big| \leqslant \int_s^t |\dot{x}_u| \, u^{\frac{\alpha}{2(1-\alpha)}} \, u^{-\frac{\alpha}{2(1-\alpha)}} \, du \\
&\leqslant \Big( \int_s^t (\dot{x}_u)^2 \, u^{\frac{\alpha}{1-\alpha}} \, du \Big)^{\frac{1}{2}} \Big( \int_s^t u^{-\frac{\alpha}{1-\alpha}} \, du \Big)^{\frac{1}{2}} \leqslant \sqrt{\eta \, \tfrac{1-\alpha}{1-2\alpha}} \, \big( t^{\frac{1-2\alpha}{1-\alpha}} - s^{\frac{1-2\alpha}{1-\alpha}} \big)^{\frac{1}{2}},
\end{aligned}
$$

which proves equicontinuity. The boundedness condition follows from this, together with the observation that $0 \geqslant x_0 \geqslant -c\eta/f(0)$. $\square$

To move our moderate deviation principle to the topology of uniform convergence on compact sets, recall the definition of the mappings $f_m$ from (19). We abbreviate

$$
\bar{Z}^{(\kappa)} := \Big( \tfrac{1}{a_\kappa} (Z_{\kappa t} - \kappa t) \, : \, t > 0 \Big).
$$

**Lemma 4.16.** *$(f_m(\bar{Z}^{(\kappa)}))_{m \in \mathbb{N}}$ are exponentially good approximations of $(\bar{Z}^{(\kappa)})$ on $\mathcal{L}(0, \infty)$.*

**Proof.** We need to verify that, denoting by $\| \cdot \|$ the supremum norm on any compact subset of $(0, \infty)$, for every $\delta > 0$,

$$
\lim_{m \to \infty} \limsup_{\kappa \to \infty} a_\kappa^{-1} b_\kappa^{-1} \log \mathbb{P}\big( \|\bar{Z}^{(\kappa)} - f_m(\bar{Z}^{(\kappa)})\| > \delta \big) = -\infty.
$$

The crucial step is to establish that, for sufficiently large $\kappa$, for all $j \geqslant 2$,

$$
\mathbb{P}\Big( \sup_{t_{j-1} \leqslant t < t_j} \big| \bar{Z}_t^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \delta \Big) \leqslant 2 \, \mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \tfrac{\delta}{2} \big). \tag{25}
$$

To verify (25) we use the stopping time $\tau := \inf\{ t \geqslant t_{j-1} \colon |\bar{Z}_t^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)}| > \delta \}$. Note that

$$
\begin{aligned}
&\mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \tfrac{\delta}{2} \big) \\
&\quad \geqslant \mathbb{P}\Big( \sup_{t_{j-1} \leqslant t < t_j} \big| \bar{Z}_t^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \delta \Big) \, \mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \tfrac{\delta}{2} \, \big| \, \tau \leqslant t_j \big),
\end{aligned}
$$

and, using Chebyshev's inequality in the last step,

$$
\mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \tfrac{\delta}{2} \, \big| \, \tau \leqslant t_j \big) \geqslant \mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{\tau}^{(\kappa)} \big| \leqslant \tfrac{\delta}{2} \, \big| \, \tau \leqslant t_j \big) \geqslant 1 - \tfrac{4}{\delta^2} \, \mathrm{Var}\big( \bar{Z}_{t_j - t_{j-1}}^{(\kappa)} \big).
$$

As this variance is of order $a_\kappa^{-2} \, \kappa^{\frac{1-2\alpha}{1-\alpha}} \, \bar{\ell}(\kappa)^{-1} \to 0$, the right hand side exceeds $\frac{1}{2}$ for sufficiently large $\kappa$, thus proving (25).

With (25) at our disposal, we observe that, for some integers $n_1 \geqslant n_0 \geqslant 2$ depending only on $m$ and the chosen compact subset of $(0, \infty)$,

$$
\begin{aligned}
\mathbb{P}\big( \|\bar{Z}^{(\kappa)} - f_m(\bar{Z}^{(\kappa)})\| > \delta \big) &\leqslant \sum_{j=n_0}^{n_1} \mathbb{P}\Big( \sup_{t_{j-1} \leqslant t < t_j} \big| \bar{Z}_t^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \delta \Big) \\
&\leqslant 2 \sum_{j=n_0}^{n_1} \mathbb{P}\big( \big| \bar{Z}_{t_j}^{(\kappa)} - \bar{Z}_{t_{j-1}}^{(\kappa)} \big| > \tfrac{\delta}{2} \big).
\end{aligned}
$$

Hence, we get

$$\limsup_{\kappa\to\infty} a_\kappa^{-1} b_\kappa^{-1} \log \mathbb{P}\big(\|\bar{Z}^{(\kappa)} - f_m(\bar{Z}^{(\kappa)})\| > \delta\big) \leqslant - \inf_{j=n_0}^{n_1} I_{[t_{j-1},t_j)}(\tfrac{\delta}{2}),$$

and the right hand side can be made arbitrarily small by making $m = \frac{1}{t_j - t_{j-1}}$ large. □

**Proof of Theorem 1.12.** We apply (Dembo and Zeitouni, 1998, Theorem 4.2.23) to transfer the large deviation principle from the topological Hausdorff space $\mathcal{P}(0,\infty)$ to the metrizable space $\mathcal{L}(0,\infty)$ using the sequence $f_m$ of continuous functions. There are two conditions to be checked for this, on the one hand that $(f_m(\bar{Z}^{(\kappa)}))_{m\in\mathbb{N}}$ are exponentially good approximations of $(\bar{Z}^{(\kappa)})$, as verified in Lemma 4.16, on the other hand that

$$\limsup_{m\to\infty} \sup_{I(x)\leqslant\eta} d\big(f_m(x), x\big) = 0,$$

for every $\eta > 0$, where $d$ denotes a suitable metric on $\mathcal{L}(0,\infty)$. This follows easily from the equicontinuity of the set $\{I(x) \leqslant \eta\}$ established in Lemma 4.15. Hence the proof is complete. □

# 5 The vertex with maximal indegree

In this section we prove Theorem 1.7 and Theorem 1.15.

## 5.1 Strong and weak preference: Proof of Theorem 1.7

The key to the proof is Proposition 5.1 which shows that, in the strong preference case, the degree of a fixed vertex can only be surpassed by a finite number of future vertices. The actual formulation of the result also contains a useful technical result for the weak preference case. Recall that $\varphi_t = \int_0^t \frac{1}{f(v)}\, dv$, and let

$$t(s) = \sup\{t \in \mathbb{S} \colon 4\varphi_t \leqslant s\}, \qquad \text{for } s \geqslant 0.$$

Moreover, we let $\varphi_\infty = \lim_{t\to\infty} \varphi_t$, which is finite exactly in the strong preference case. In this case $t(s) = \infty$ eventually.

**Proposition 5.1.** *For any fixed $\eta > 0$, almost surely only finitely many of the events*

$$A_s := \big\{\exists t' \in [s, t(s)) \cap \mathbb{T} : Z[s, t'] \geqslant t' - \eta\big\}, \qquad \textit{for } s \in \mathbb{T},$$

*occur.*

For the proof we identify a family of martingales and then apply the concentration inequality for martingales, Lemma A.3. For $s \in \mathbb{T}$, let $(\bar{T}_u^s)_{u\in\mathbb{S}}$ be given by $\bar{T}_u^s = u - T_s[0,u)$, where $T_s[u,v)$ is the time spent by the process $Z[s,\cdot\,]$ in the interval $[u,v)$.
The following lemma is easy to verify.

**Lemma 5.2.** *Let $(t_i)_{i\in\mathbb{Z}_+}$ be a strictly increasing sequence of nonnegative numbers with $t_0 = 0$ and $\lim_{i\to\infty} t_i = \infty$. Moreover, assume that $\lambda > 0$ is fixed such that $\lambda\,\Delta t_i := \lambda\,(t_i - t_{i-1}) \leqslant 1$, for all $i \in \mathbb{N}$, and consider a discrete random variable $X$ with*

$$\mathbb{P}\big(X = t_i\big) = \lambda\Delta t_i \prod_{j=1}^{i-1}(1-\lambda\Delta t_j) \qquad \textit{for } i \in \mathbb{N}.$$

*Then*

$$\mathbb{E}[X] = \frac{1}{\lambda} \qquad \textit{and} \qquad \operatorname{var}(X) \leqslant \frac{1}{\lambda^2}.$$

With this at hand, we can identify the martingale property of $(\bar{T}^s_u)_{u\in\mathbb{S}}$.

**Lemma 5.3.** *For any $s \in \mathbb{S}$, the process $(\bar{T}^s_u)_{u\in\mathbb{S}}$ is a martingale with respect to the natural filtration $(\mathcal{G}_u)$. Moreover, for two neighbours $u < u_+$ in $\mathbb{S}$, one has*

$$\operatorname{var}\big(\bar{T}^s_{u_+}\,|\,\mathcal{G}_u\big) \leqslant \frac{1}{\bar{f}(u)^2}.$$

**Proof.** Fix two neighbours $u < u_+$ in $\mathbb{S}$ and observe that given $\mathcal{G}_u$ (or given the entry time $T_s[0,u) + s$ into state $u$) the distribution of $T_s[u]$ is as in Lemma 5.2 with $\lambda = \bar{f}(u)$. Thus the lemma implies that

$$\mathbb{E}[\bar{T}^s_{u+}|\mathcal{G}_u] = \bar{T}^s_u + \frac{1}{f(u)} - \mathbb{E}[T_s[u]\,|\,\mathcal{G}_u] = \bar{T}^s_u$$

so that $(\bar{T}^s_u)$ is a martingale. The variance estimate of Lemma 5.2 yields the second assertion. $\square$

**Proof of Proposition 5.1.** We fix $\eta \geqslant 1/f(0)$ and $u_0 \in \mathbb{S}$ with $\bar{f}(u_0) \geqslant 2$. We consider $\mathbb{P}(A_s)$ for sufficiently large $s \in \mathbb{T}$. More precisely, $s$ needs to be large enough such that $t(s) \geqslant u_0$ and $s - \eta - u_0 \geqslant \sqrt{s/2}$. We denote by $\sigma$ the first time $t$ in $\mathbb{T}$ for which $Z[s,t] \geqslant t - \eta$, if such a time exists, and set $\sigma = \infty$ otherwise.

We now look at realizations for which $\sigma \in [s, t(s))$ or, equivalently, $A_s$ occurs. We set $\nu = Z[s,\sigma]$. Since the jumps of $Z[s,\cdot]$ are bounded by $1/f(0)$ we conclude that

$$\nu \leqslant \sigma - \eta + 1/f(0) \leqslant \sigma.$$

Conversely, $T_s[0,\nu) + s$ is the entry time into state $\nu$ and thus equal to $\sigma$; therefore,

$$\nu = Z[s,\sigma] \geqslant T_s[0,\nu) + s - \eta,$$

and thus $\bar{T}^s_\nu = \nu - T_s[0,\nu) \geqslant s - \eta$. Altogether, we conclude that

$$A_s \subset \big\{\exists u \in [0,t(s)) \cap \mathbb{S} \colon \bar{T}^s_u \geqslant s - \eta\big\}.$$

By Lemma 5.3 the process $(\bar{T}^s_u)_{u\in\mathbb{S}}$ is a martingale. Moreover, for consecutive elements $u < u_+$ of $\mathbb{S}$ that are larger than $u_0$, one has

$$\operatorname{var}(\bar{T}^s_{u+}|\mathcal{G}_u) = \frac{1}{\bar{f}(u)^2}, \qquad \bar{T}^s_{u+} - \bar{T}^s_u \leqslant \frac{1}{\bar{f}(u)} \leqslant \frac{1}{2}, \qquad \text{and} \qquad \bar{T}^s_{u_0} \leqslant u_0.$$

Now we apply the concentration inequality, Lemma A.3, and obtain, writing $\lambda_s = s - \eta - u_0 - \sqrt{2\varphi_{t(s)}} \geqslant 0$, that

$$\begin{aligned}\mathbb{P}(A_s) &\leqslant \mathbb{P}\Big(\sup_{u\in[0,t(s))\cap\mathbb{S}} \bar{T}^s_u \geqslant s-\eta\Big)\\ &\leqslant \mathbb{P}\Big(\sup_{u\in[u_0,t(s))\cap\mathbb{S}} \bar{T}^s_u - \bar{T}^s_{u_0} \geqslant s-\eta-u_0\Big) \leqslant 2\exp\Big(-\frac{\lambda_s^2}{2(\varphi_{t(s)}+\lambda_s/6)}\Big),\end{aligned}$$

where we use that

$$\sum_{u\in\mathbb{S}\cap[0,t(s))} \frac{1}{\bar{f}(u)^2} = \varphi_{t(s)}.$$

As $\varphi_{t(s)} \leqslant s/4$, we obtain $\limsup -\frac{1}{s}\log\mathbb{P}(A_s) \geqslant \frac{6}{5}$. Denoting by $\iota(t) = \max[0,t]\cap\mathbb{T}$, we finally get that

$$\sum_{s\in\mathbb{T}} \mathbb{P}(A_s) \leqslant \int_0^\infty e^s\,\mathbb{P}(A_{\iota(s)})\,ds < \infty,$$

so that by Borel-Cantelli, almost surely, only finitely many of the events $(A_s)_{s\in\mathbb{T}}$ occur. □

**Proof of Theorem 1.7.** We first consider the weak preference case and fix $s\in\mathbb{T}$. Recall that $(Z[s,t]-(t-s))_{t\geqslant s}$ and $(Z[0,t]-t)_{t\geqslant 0}$ are independent and satisfy functional central limit theorems (see Theorem 1.6). Thus $(Z[s,t]-Z[0,t])_{t\geqslant s}$ also satisfies a central limit theorem, i.e. an appropriately scaled version converges weakly to the Wiener process. Since the Wiener process changes its sign almost surely for arbitrarily large times, we conclude that $Z[s,t]$ will be larger, respectively smaller, than $Z[0,t]$ for infinitely many time instances. Therefore, $s$ is not a persistent hub, almost surely. This proves the first assertion.

In the strong preference case recall that $\varphi_\infty < \infty$. For fixed $\eta > 0$, almost surely, only finitely many of the events $(A_s)_{s\in\mathbb{T}}$ occur, by Proposition 5.1. Recalling that $Z[0,t]-t$ has a finite limit, we thus get that almost surely only finitely many degree evolutions overtake the one of the first node. It remains to show that the limit points of $(Z[s,t]-t)$ for varying $s\in\mathbb{T}$ are almost surely distinct. But this is an immediate consequence of Proposition 2.2. □

## 5.2 The typical evolution of the hub: Proof of Theorem 1.15

From now on we assume that the attachment rule $f$ is regularly varying with index $\alpha < \frac{1}{2}$, and we represent $f$ and $\bar{f}$ as

$$f(u) = u^\alpha \ell(u) \quad\text{ and }\quad \bar{f}(u) = u^{\frac{\alpha}{1-\alpha}}\bar{\ell}(u) \qquad \text{for } u>0.$$

Moreover, we fix

$$a_\kappa = \kappa^{\frac{1-2\alpha}{1-\alpha}}\bar{\ell}(\kappa)^{-1}.$$

For this choice of $(a_\kappa)$ the moderate deviation principle, Theorem 1.12, leads to the speed $(a_\kappa)$, in other words the magnitude of the deviation and the speed coincide. The proof of Theorem 1.15 is based on the following lemma.

**Lemma 5.4.** *Fix $0\leqslant u<v$ and define $\mathbb{I}_\kappa$ as $\mathbb{I}_\kappa = \mathbb{T}\cap[a_\kappa u, a_\kappa v)$. Then, for all $\varepsilon>0$,*

$$\lim_{\kappa\to\infty} \mathbb{P}\Big(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] \in \kappa + a_\kappa\Big[-v+\sqrt{\tfrac{2-2\alpha}{1-2\alpha}v}-\varepsilon, -u+\sqrt{\tfrac{2-2\alpha}{1-2\alpha}v}+\varepsilon\Big]\Big) = 1.$$

**Proof.** Our aim is to analyze the random variable $\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa]$ for large $\kappa$. We fix $\zeta \geqslant -u$ and observe that

$$\mathbb{P}(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] < \kappa + a_\kappa\zeta) = \prod_{s\in\mathbb{I}_\kappa} \mathbb{P}(Z[s,\kappa] < \kappa + a_\kappa\zeta)$$
$$\begin{cases} \leqslant \mathbb{P}(Z[s_{\max},\kappa] < \kappa + a_\kappa\zeta)^{\#\mathbb{I}_\kappa} = \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} > \kappa)^{\#\mathbb{I}_\kappa} \\ \geqslant \mathbb{P}(Z[s_{\min},\kappa] < \kappa + a_\kappa\zeta)^{\#\mathbb{I}_\kappa} = \mathbb{P}(T_{s_{\min}}[0,\kappa + a_\kappa\zeta) + s_{\min} > \kappa)^{\#\mathbb{I}_\kappa}, \end{cases} \tag{26}$$

where $s_{\min}$ and $s_{\max}$ denote the minimal and maximal element of $\mathbb{I}_\kappa$.
Next, we observe that $\lim_{\kappa\to\infty} s_{\max}/a_\kappa = v$ and $\lim_{\kappa\to\infty} s_{\min}/a_\kappa = u$. Consequently, we can deduce from the moderate deviation principle, Lemma 4.12, together with Lemma 4.2, that

$$\begin{aligned} \log \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} \leqslant \kappa) &= \log \mathbb{P}\Big(\frac{T_{s_{\max}}[0,\kappa + a_\kappa\zeta) - \kappa - a_\kappa\zeta}{a_\kappa} \leqslant -\frac{s_{\max}}{a_\kappa} - \zeta\Big) \\ &\sim -a_\kappa I_{[0,1)}(-v-\zeta) = -a_\kappa \frac{1}{2}\frac{1-2\alpha}{1-\alpha}(v+\zeta)^2 \end{aligned} \tag{27}$$

and analogously that

$$\log \mathbb{P}(T_{s_{\min}}[0,\kappa + a_\kappa\zeta) + s_{\min} \leqslant \kappa) \sim -a_\kappa \frac{1}{2}\frac{1-2\alpha}{1-\alpha}(u+\zeta)^2.$$

Next we prove that $\mathbb{P}(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] < \kappa + a_\kappa\zeta)$ tends to 0 when $\zeta < -v + \sqrt{\frac{2-2\alpha}{1-2\alpha} v}$.
If $\zeta < -u$, then the statement is trivial since by the moderate deviation principle $\mathbb{P}(Z[s_{\min},\kappa] < \kappa + a_\kappa\zeta)$ tends to zero. Thus we can assume that $\zeta \geqslant -u$. By (26) one has

$$\mathbb{P}\big(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] < \kappa + a_\kappa\zeta\big) \leqslant \exp\big\{\#\mathbb{I}_\kappa \log\big(1 - \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} \leqslant \kappa)\big)\big\}.$$

and it suffices to show that the term in the exponential tends to $-\infty$ in order to prove the assertion. The term satisfies

$$\begin{aligned} &\#\mathbb{I}_\kappa \log\big(1 - \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} \leqslant \kappa)\big) \\ &\quad \sim -\#\mathbb{I}_\kappa\, \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} \leqslant \kappa) \\ &\quad = -\exp\Big\{a_\kappa \underbrace{\Big[\frac{1}{a_\kappa}\log \#\mathbb{I}_\kappa + \frac{1}{a_\kappa}\log \mathbb{P}(T_{s_{\max}}[0,\kappa + a_\kappa\zeta) + s_{\max} \leqslant \kappa)\Big]}_{=:c_\kappa}\Big\}. \end{aligned}$$

Since $\frac{1}{a_\kappa}\log \#\mathbb{I}_\kappa$ converges to $v$, we conclude with (27) that

$$\lim_{\kappa\to\infty} c_\kappa = v - \tfrac{1-2\alpha}{2-2\alpha}(v+\zeta)^2.$$

Now elementary calculus implies that the limit is bigger than 0 by choice of $\zeta$. This implies the first part of the assertion.
It remains to prove that $\mathbb{P}(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] < \kappa + a_\kappa\zeta)$ tends to 1 for $\zeta > -u + \sqrt{\frac{2-2\alpha}{1-2\alpha} v}$. Now

$$\mathbb{P}\big(\max_{s\in\mathbb{I}_\kappa} Z[s,\kappa] < \kappa + a_\kappa\zeta\big) \geqslant \exp\Big\{\#\mathbb{I}_\kappa \log\big(1 - \mathbb{P}(T_{s_{\min}}[0,\kappa + a_\kappa\zeta) + s_{\min} \leqslant \kappa)\big)\Big\}$$

and it suffices to show that the expression in the exponential tends to 0. As above we conclude that

$$\begin{aligned}\#\mathbb{I}_\kappa \log(1-\mathbb{P}(T_{s_{\min}}[0,\kappa+a_\kappa\zeta)+s_{\min}\leqslant\kappa))\\ \sim -\exp\big(a_\kappa \underbrace{\Big[\frac{1}{a_\kappa}\log\#\mathbb{I}_\kappa+\frac{1}{a_\kappa}\log\mathbb{P}(T_{s_{\min}}[0,\kappa+a_\kappa\zeta)+s_{\min}\leqslant\kappa)\Big]}_{=:c_\kappa}\big).\end{aligned}$$

We find convergence

$$\lim_{\kappa\to\infty} c_\kappa = v-\tfrac{1-2\alpha}{2-2\alpha}\,(u+\zeta)^2$$

and (as elementary calculus shows) the limit is negative by choice of $\zeta$. □

For $s\in\mathbb{T}$ and $\kappa>0$ we denote by $\bar Z^{(s,\kappa)}=(\bar Z^{(s,\kappa)}_t)_{t\,\geqslant\,0}$ the random evolution given by

$$\bar Z^{(s,\kappa)}_t=\frac{Z[s,s+\kappa t]-\kappa t}{a_\kappa}.$$

Moreover, we let

$$z=(z_t)_{t\geqslant 0}=\Big(\tfrac{1-\alpha}{1-2\alpha}\big(t^{\frac{1-2\alpha}{1-\alpha}}\wedge 1\big)\Big)_{t\,\geqslant\,0}.$$

**Proof of Theorem 1.15.** *1st Part:* By Lemma 5.4 the maximal indegree is related to the unimodal function $h$ defined by

$$h(u)=-u+\sqrt{\tfrac{2-2\alpha}{1-2\alpha}u},\qquad \text{for } u\geqslant 0.$$

$h$ attains its unique maximum in $u_{\max}=\frac12\frac{1-\alpha}{1-2\alpha}$ and $h(u_{\max})=u_{\max}$. We fix $c>4\frac{1-\alpha}{1-2\alpha}$, let $\zeta=\max[h(u_{\max})-h(u_{\max}\pm\varepsilon)]$ and decompose the set $[0,u_{\max}-\varepsilon)\cup[u_{\max}+\varepsilon,c)$ into finitely many disjoint intervals $[u_i,v_i)$ $i\in\mathbb{J}$, with mesh smaller than $\zeta/3$. Then for the hub $s^*_\kappa$ at time $\kappa>0$ one has

$$\begin{aligned}&\mathbb{P}(s^*_\kappa\in a_\kappa[u_{\max}-\varepsilon,u_{\max}+\varepsilon))\\ &\quad\geqslant \mathbb{P}\Big(\max_{s\in a_\kappa[u_{\max}-\varepsilon,u_{\max})\cap\mathbb{T}} Z[s,\kappa]\geqslant\kappa+a_\kappa(h(u_{\max})-\zeta/3)\Big)\\ &\qquad\times\prod_{i\in\mathbb{J}}\mathbb{P}\Big(\max_{s\in a_\kappa[u_i,v_i)\cap\mathbb{T}} Z[s,\kappa]\leqslant\kappa+a_\kappa(h(v_i)+\zeta/2)\Big)\\ &\qquad\times\mathbb{P}\Big(\max_{s\in[c\,a_\kappa,\infty)\cap\mathbb{T}} Z[s,\kappa]\leqslant\kappa\Big).\end{aligned}\tag{28}$$

By Lemma 5.4 the terms in the first two lines on the right terms converge to 1. Moreover, by Proposition 5.1 the third term converges to 1, if for all sufficiently large $\kappa$ and $\kappa_+=\min[\kappa,\infty)\cap\mathbb{S}$, one has $4\varphi_{\kappa_+}\leqslant ca_\kappa$. This is indeed the case, since one has $\kappa_+\leqslant\kappa+f(0)^{-1}$ so that by Lemma A.1, $4\varphi_{\kappa_+}\sim 4\frac{1-\alpha}{1-2\alpha}\,a_\kappa$. The statement on the size of the maximal indegree is now an immediate consequence of Lemma 5.4.

*2nd Part:* We now prove that (an appropriately scaled version of) the evolution of a hub typically lies in an open neighbourhood around $z$.

Let $U$ denote an open set in $\mathcal{L}(0,\infty)$ that includes $z$ and denote by $U^c$ its complement in $\mathcal{L}(0,\infty)$. Furthermore, we set

$$A_\varepsilon = \Big\{x \in \mathcal{L}(0,\infty) : \max_{t\in[\frac{1}{2},1]} x_t \geqslant 2(u_{\max} - \varepsilon)\Big\}$$

for $\varepsilon \geqslant 0$. We start by showing that $z$ is the unique minimizer of $I$ on the set $A_0$. Indeed, applying the inverse Hölder inequality gives, for $x \in A_0$ with finite rate $I(x)$,

$$I(x) \geqslant \tfrac{1}{2}\int_0^1 \dot{x}_t^2\, t^{\frac{\alpha}{1-\alpha}}\, dt \geqslant \tfrac{1}{2}\Big(\int_0^1 |\dot{x}_t|\, dt\Big)^2\Big(\int_0^1 t^{-\frac{\alpha}{1-\alpha}}\, dt\Big)^{-1} \geqslant \tfrac{1}{2}\tfrac{1-\alpha}{1-2\alpha} = u_{\max} = I(z).$$

Moreover, one of the three inequalities is a strict inequality when $x \neq z$. Recall that, by Lemma 4.15, $I$ has compact level sets. We first assume that one of the entries in $U^c \cap A_0$ has finite rate $I$. Since $U^c \cap A_0$ is closed, we conclude that $I$ attains its infimum on $U^c \cap A_0$. Therefore,

$$I(U^c \cap A_0) := \inf\{I(x)\colon x \in U^c \cap A_0\} > I(z) = u_{\max}.$$

Conversely, using again compactness of the level sets, gives

$$\lim_{\varepsilon\downarrow 0} I(U^c \cap A_\varepsilon) = I(U^c \cap A_0).$$

Therefore, there exists $\varepsilon > 0$ such that $I(U^c \cap A_\varepsilon) > I(z)$. Certainly, this is also true if $U^c$ contains no element of finite rate.

From the moderate deviation principle, Theorem 1.12, together with the uniformity in $s$, see Proposition 4.4, we infer that

$$\limsup_{\kappa\to\infty} \frac{1}{a_\kappa} \max_{s\in\mathbb{T}} \log \mathbb{P}\big(\bar{Z}^{(s,\kappa)} \in U^c \cap A_\varepsilon\big) \leqslant -I(U^c \cap A_\varepsilon) < -I(z). \tag{29}$$

It remains to show that $\mathbb{P}(\bar{Z}^{s^*_\kappa,\kappa} \in U^c)$ converges to zero. For $\varepsilon > 0$ and sufficiently large $\kappa$,

$$\begin{aligned}
\mathbb{P}(\bar{Z}^{s^*_\kappa,\kappa} \in U^c) \leqslant{}& \mathbb{P}(s^*_\kappa \notin a_\kappa[u_{\max} - \varepsilon, u_{\max} + \varepsilon]) \\
&+ \mathbb{P}\big(\max_{t\in[\frac{1}{2},1]} \bar{Z}_t^{s^*_\kappa,\kappa} \leqslant 2(u_{\max} - \varepsilon), s^*_\kappa \in a_\kappa[u_{\max} - \varepsilon, u_{\max} + \varepsilon]\big) \\
&+ \mathbb{P}\big(\bar{Z}^{s^*_\kappa,\kappa} \in U^c, \max_{t\in[\frac{1}{2},1]} \bar{Z}_t^{s^*_\kappa,\kappa} \geqslant 2(u_{\max} - \varepsilon), s^*_\kappa \in a_\kappa[u_{\max} - \varepsilon, u_{\max} + \varepsilon]\big).
\end{aligned}$$

By the first part of the proof the first and second term in the last equation tend to 0 for any $\varepsilon > 0$. The last term can be estimated as follows

$$\begin{aligned}
\mathbb{P}\big(\bar{Z}^{s^*_\kappa,\kappa} \in U^c, \bar{Z}_t^{s^*_\kappa,\kappa} \geqslant 2(u_{\max} - \varepsilon), s^*_\kappa \in a_\kappa[u_{\max} - \varepsilon, u_{\max} + \varepsilon]\big)& \\
\leqslant \sum_{s\in\mathbb{T}\cap a_\kappa[u_{\max}-\varepsilon, u_{\max}+\varepsilon]} \mathbb{P}(\bar{Z}^{(s,\kappa)} \in U^c \cap A_\varepsilon).&
\end{aligned} \tag{30}$$

Moreover, $\log \#\big(\mathbb{T} \cap a_\kappa[u_{\max} - \varepsilon, u_{\max} + \varepsilon]\big) \sim a_\kappa(u_{\max} + \varepsilon)$. Since, for sufficiently small $\varepsilon > 0$, we have $I(U^c \cap A_\varepsilon) > u_{\max} + \varepsilon$ we infer from (29) that the sum in (30) goes to zero. □

# A Appendix

## A.1 Regularly varying attachment rules

In the following we assume that $f\colon [0,\infty) \to (0,\infty)$ is a regularly varying attachment rule with index $\alpha < 1$, and represent $f$ as $f(u) = u^\alpha \ell(u)$, for $u > 0$, with a slowly varying function $\ell$.

**Lemma A.1.** *1. One has*

$$\Phi(u) \sim \frac{1}{1-\alpha}\frac{u^{1-\alpha}}{\ell(u)}$$

*as $u$ tends to infinity and $\bar f$ admits the representation*

$$\bar f(u) = f \circ \Phi^{-1}(u) = u^{\frac{\alpha}{1-\alpha}}\bar\ell(u), \qquad \text{for } u > 0,$$

*where $\bar\ell$ is again a slowly varying function.*

*2. If additionally $\alpha < \frac{1}{2}$, then*

$$\varphi_u = \int_0^u \frac{1}{\bar f(u)}\,du \sim \frac{1-\alpha}{1-2\alpha}\frac{u^{\frac{1-2\alpha}{1-\alpha}}}{\bar\ell(u)}.$$

**Proof.** The results follow from the theory of regularly variation, and we briefly quote the relevant results taken from Bingham et al. (1987). The asymptotic formula for $\Phi$ is an immediate consequence of Karamata's theorem, Theorem 1.5.11. Moreover, by Theorem 1.5.12, the inverse of $\Phi$ is regularly varying with index $(1-\alpha)^{-1}$ so that, by Proposition 1.5.7, the composition $\bar f = f \circ \Phi^{-1}$ is regularly varying with index $\frac{\alpha}{1-\alpha}$. The asymptotic statement about $\varphi$ follows again by Karamata's theorem. □

**Remark A.2.** In the particular case where $f(u) \sim cu^\alpha$, we obtain

$$\Phi(u) \sim \tfrac{1}{c(1-\alpha)}\, u^{1-\alpha},\ \Phi^{-1}(u) \sim (c(1-\alpha)u)^{\frac{1}{1-\alpha}}$$

and

$$\bar f(u) \sim c^{\frac{1}{1-\alpha}}\big((1-\alpha)u\big)^{\frac{\alpha}{1-\alpha}}.$$

## A.2 Two concentration inequalities for martingales

**Lemma A.3.** *Let $(M_n)_{n\in\mathbb{Z}_+}$ be a martingale for its canonical filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ with $M_0 = 0$. We assume that there are deterministic $\sigma_n \in \mathbb{R}$ and $M < \infty$ such that almost surely*

- $\mathrm{var}(M_n|\mathcal{F}_{n-1}) \leqslant \sigma_n^2$ *and*
- $M_n - M_{n-1} \leqslant M$.

*Then, for any $\lambda > 0$ and $m \in \mathbb{N}$,*

$$\mathbb{P}\Big(\sup_{n\leqslant m} M_n \geqslant \lambda + \sqrt{2\sum_{n=1}^m \sigma_n^2}\Big) \leqslant 2\exp\Big(-\frac{\lambda^2}{2(\sum_{n=1}^m \sigma_n^2 + M\lambda/3)}\Big).$$

**Proof.** Let $\tau$ denote the first time $n \in \mathbb{N}$ for which $M_n \geqslant \lambda + \sqrt{2\sum_{n=1}^m \sigma_n^2}$. Then

$$\mathbb{P}(M_m \geqslant \lambda) \geqslant \sum_{n=1}^m \mathbb{P}(\tau = n)\, \mathbb{P}\Big(M_m - M_n \geqslant -\sqrt{2\sum_{i=1}^m \sigma_i^2}\,\Big|\, \tau = n\Big).$$

Next, observe that $\mathrm{var}(M_m - M_n \,|\, \tau = n) \leqslant \sum_{i=n+1}^m \sigma_i^2$ so that by Chebyshev's inequality

$$\mathbb{P}\Big(M_m - M_n \geqslant -\sqrt{2\sum_{i=1}^m \sigma_i^2}\,\Big|\, \tau = n\Big) \geqslant 1/2.$$

On the other hand, a concentration inequality of Azuma type gives

$$\mathbb{P}(M_m \geqslant \lambda) \leqslant \exp\Big(-\frac{\lambda^2}{2(\sum_{n=1}^m \sigma_n^2 + M\lambda/3)}\Big)$$

(see for instance Chung and Lu (2006), Theorem 2.21). Combining these estimates immediately proves the assertion of the lemma. □

Similarly, one can use the classical Azuma-Hoeffding inequality to prove the following concentration inequality.

**Lemma A.4.** *Let $(M_n)_{n\in\mathbb{Z}_+}$ be a martingale such that almost surely $|M_n - M_{n-1}| \leqslant c_n$ for given sequence $(c_n)_{n\in\mathbb{N}}$. Then for any $\lambda > 0$ and $m \in \mathbb{N}$*

$$\mathbb{P}\Big(\sup_{n\leqslant m} |M_n - M_0| \geqslant \lambda + \sqrt{2\sum_{n=1}^m c_n^2}\Big) \leqslant 4\exp\Big(-\frac{\lambda^2}{2\sum_{n=1}^m c_n^2}\Big).$$

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## References

Barabási, A.-L. and R. Albert (1999). Emergence of scaling in random networks. *Science 286*(5439), 509–512.

Bingham, N. H., C. M. Goldie, and J. L. Teugels (1987). *Regular variation*, Volume 27 of *Encyclopedia of Mathematics and its Applications.* Cambridge: Cambridge University Press.

Bollobás, B., O. Riordan, J. Spencer, and G. Tusnády (2001). The degree sequence of a scale-free random graph process. *Random Structures Algorithms 18*, 279–290.

Chernoff, H. (1981). A note on an inequality involving the normal distribution. *Ann. Probab. 9*(3), 533–535.

Chung, F. and L. Lu (2006). *Complex graphs and networks*, Volume 107 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC.

Dembo, A. and O. Zeitouni (1998). *Large deviations techniques and applications* (Second ed.), Volume 38 of *Applications of Mathematics*. New York: Springer-Verlag.

Jacod, J. and A. N. Shiryaev (2003). *Limit theorems for stochastic processes* (Second ed.), Volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Berlin: Springer-Verlag.

Krapivsky, P. and S. Redner (2001). Organization of growing random networks. *Phys. Rev. E 63*, Paper 066123.

Móri, T. F. (2002). On random trees. *Studia Sci. Math. Hungar. 39*, 143–155.

Móri, T. F. (2005). The maximum degree of the Barabási-Albert random tree. *Combin. Probab. Comput. 14*, 339–348.

Oliveira, R. and J. Spencer (2005). Connectivity transitions in networks with super-linear preferential attachment. *Internet Math. 2*, 121–163.

Rudas, A., B. Tóth, and B. Valkó (2007). Random trees and general branching processes. *Random Structures and Algorithms 31*, 186–202.